ON THE INDEX-$r$-FREE SEQUENCES OVER FINITE CYCLIC GROUPS

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Abstract. Let $C_n$ be a finite cyclic group of order $n \geq 2$. Every sequence $S$ over $C_n$ can be written in the form $S = (n_1g), \ldots, (n_lg)$ where $g \in C_n$ and $n_1, \ldots, n_l \in [1, \text{ord}(g)]$, and the index $\text{ind}(S)$ of $S$ is defined as the minimum of $(n_1 + \ldots + n_l)/\text{ord}(g)$ over all $g \in C_n$ with $\text{ord}(g) = n$. Let $d > 1$ and $r \geq 1$ be any fixed integers. We prove that, for every sufficiently large integer $n$ divisible by $d$, there exists a sequence $S$ over $C_n$ of length $|S| \geq n + n/d + O(\sqrt{n})$ having no subsequence $T$ of index $\text{ind}(T) \in [1, r]$, which has substantially improved the previous results in this direction.

1. Introduction and Main Results

Throughout this paper, let $C_n$ be an additively written finite cyclic group of order $|C_n| = n$, where $n \in \mathbb{Z}$ with $n > 1$. By a sequence $S$ of length $|S| = \ell$ over $C_n$ we mean an unordered sequence with $\ell$ terms from $C_n$ and the repetition of terms is allowed. We call $S$ a zero-sum sequence if the sum of $S$ is zero. We let $\mathbb{Z}$ denote the integers, and $\mathbb{R}$ the real numbers. Given real numbers $a, b \in \mathbb{R}$, we use $[a, b] := \{u : u \in \mathbb{Z}, a \leq u \leq b\}$ to denote all integers between $a$ and $b$. Recall that the index of a sequence $S$ is defined as follows.

Definition 1.1. For a sequence $S = (n_1g) \cdot \ldots \cdot (n_lg)$ over $C_n$, where $n_1, \ldots, n_l \in [1, n]$ and $g \in C_n$ with $\text{ord}(g) = |C_n|$, we set

$$\|S\|_g = \frac{n_1 + \ldots + n_l}{n},$$

and the index of $S$ is defined by

$$\text{ind}(S) = \min\{\|S\|_g \mid g \in C_n \text{ with } \text{ord}(g) = |C_n|\}.$$

The index of a sequence is a crucial invariant in the investigation of zero-sum sequences over cyclic groups. It was first addressed by Lemke and Kleitman ([9]), used as a key tool by Geroldinger ([1], page 736), and then investigated by Gao [3] in a systematical way. And it has found a lot of attention in recent years (see [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16]). If $S$ is a minimal zero-sum sequence, then $|S| \leq 3$, as well as $|S| \geq \left\lfloor \frac{n}{2} \right\rfloor + 2$, implies that $\text{ind}(S) = 1$ (see [1], [2], [11]).

An important open problem (at the end of [5]) is to determine the maximum length of sequences over $C_n$ without index 1 subsequences. Clearly, $S$ is a zero-sum sequence if and only if $\text{ind}(S)$ is an integer by definition 1.1. Hence we introduce the definitions of $t_r(n)$ and index-$r$-free sequences.

Definition 1.2. Let $r$ be a positive integer, denote by $t_r(n)$ the smallest integer $\ell$ such that every sequence $S$ over $C_n$ of length $|S| \geq \ell$ has a zero-sum subsequence $T$ with $\text{ind}(T) \in [1, r]$. 
**Theorem 1.5.** Given any fixed integers $t_k$ and prove that $d > d^2(d^3 - d^2 + d + 1)$, where $d \in \mathbb{Z}$ with $d > 1$.

In this paper we give longer general structures (Theorem 1.4) to the conjecture of Lemke and Kleitman, and prove that $t_1(n) \geq n + \frac{n}{d} - 1$, where $d \in \mathbb{Z}$ and $d > 1$ (Theorem 1.5). It is a greater lower bound of $t_1(n)$ than before, and we conjecture that it is the best possible bound when $n$ is big enough. Furthermore, we promote the index 1 free sequences to index r free sequences, and show that $t_r(n) \geq n + \frac{n}{d} + O(\sqrt{n})$ for every sufficiently large integer $n$ divisible by $d$, where constant $r \in \mathbb{Z}$ with $r \geq 2$. Here are our main results.

**Theorem 1.4.** Let $d, n$ be any integers with $1 < d|n$ and $n > d^2$, and $g \in C_n$ with $\text{ord}(g) = n$. For every integer $r \in \{1, \frac{n}{d}\}$ and $k \in \left[0, \log_d n - 2\right)$, let

$$S = \prod_{(i,j) \in A} \left( (im + d^j) g \right)^{\left\lfloor \frac{m}{d^j} \right\rfloor} - (d^r-1)d^{k-j-1},$$

is an index-$r$-free sequence, where $m = \frac{n}{d}$ and $A = [1, d-1] \times [0, k] \cup \{(0,0)\}$.

**Theorem 1.5.** Given any fixed integers $d > 1$ and $r \geq 1$, for every sufficiently large integer $n$ with $d|n$, there exists an index-$r$-free sequence $S$ over $C_n$ such that $|S| \geq n + \frac{n}{d} + O(\sqrt{n})$.

In the following sections we provide the preliminaries and the proofs of Theorem 1.4 and Theorem 1.5. We end the paper with a further conjecture and an open problem.

### 2. Notations and Preliminaries

We let $n$ and $d$ be any integers with $1 < d|n$ and $n > d^2$, and let $g \in C_n$ with $\text{ord}(g) = n$. For every integer $r \in \left[1, \frac{n}{d}\right]$ and $k \in \left[0, \log_d n - 2\right)$, let a sequence

$$S = \prod_{(i,j) \in A} \left( (im + d^j) g \right)^{\left\lfloor \frac{m}{d^j} \right\rfloor} - (d^r-1)d^{k-j-1},$$

where $m = \frac{n}{d}$ and $A = [1, d-1] \times [0, k] \cup \{(0,0)\}$.

Let $T$ be a subsequence of $S$ and $t_{ij} \in \mathbb{Z}$ be the multiplicity of $(im + d^j)g$ in $T$, where $(i,j) \in A$. If $(im + d^j)g \notin T$, we set $t_{ij} = 0$. That is,

$$T = \prod_{(i,j) \in A} \left( (im + d^j) g \right)^{t_{ij}} \subset S,$$
where

\[ 0 \leq t_{ij} \leq \left\lfloor \frac{m}{d^j} \right\rfloor - (dr - 1)d^{k-j} - 1. \]

We set \( \text{ind}(T) = \|T\|_{g_1} \), where \( g_1 \in C_n \) with \( \langle g_1 \rangle = C_n \). And we set \( g = hg_1 \), where \( h \in [1, n-1] \) with \( \gcd(h, n) = 1 \). Then

\[ T = \prod_{(i, j) \in A} (im + d^j)_{h_1}^{t_{ij}}, \]

and

\[ n \| T \|_{g_1} = \sum_{(i, j) \in A} t_{ij} \left\lfloor (im + d^j)h \right\rfloor_n, \]

where \( \left\lfloor w \right\rfloor_n \) denotes the least positive residue of \( w \in \mathbb{Z} \) modulo \( n > 0 \). We fix the notation concerning sequences over \( C_n \). And let

\[ B = \left\{ (i, j) \in A \mid 0 < \left\lfloor (im + d^j)h \right\rfloor_n < m \right\}, \]

and

\[ C = \left\{ (i, j) \in A \mid m < \left\lfloor (im + d^j)h \right\rfloor_n < n \right\}. \]

By next lemma we split \( A \) into two parts.

**Lemma 2.1.** \( B \cup C = A \).

**Proof.** For every \( (i, j) \in A \), combining \( A = [1, d-1] \times [0, k] \cup \{(0, 0)\}, r \in [1, \frac{n}{d^k}] \) with \( k \in \left[0, \log_{d} \frac{n}{d} - 2\right)\), we derive \( 0 < d^j < m \). Then by \( \gcd(h, n) = 1 \) and \( dm = n \), we have \( 0 < \left\lfloor (im + d^j)h \right\rfloor_n < n \) and \( \left\lfloor (im + d^j)h \right\rfloor_n \neq m \) for every \( (i, j) \in A \). Then by the definitions of \( B \) and \( C \), we have \( B \cup C = A \).

**Lemma 2.2.** For any integer \( j \in [0, k] \), we have

\[ \left\{ \left\lfloor (im + d^j)h \right\rfloor_n \mid i \in [0, d-1] \right\} = \left\{ im + \left\lfloor hd^j \right\rfloor_m \mid i \in [0, d-1] \right\}, \]

and there exists only one element \( i_0 \in [0, d-1] \) such that \( 0 < \left\lfloor (i_0m + d^j)h \right\rfloor_n < m \).

**Proof.** By

\[ \left\lfloor (im + d^j)h \right\rfloor_n \left\lfloor m \right\rfloor = \left\lfloor hd^j \right\rfloor_m, \]

we have

\[ \left\{ \left\lfloor (im + d^j)h \right\rfloor_n \mid i \in [0, d-1] \right\} \subseteq \left\{ im + \left\lfloor hd^j \right\rfloor_m \mid i \in \mathbb{Z} \right\}. \]

For any \( j \in [0, k] \), by the relevant definitions we have \( 0 < d^j < m \), then \( 0 < \left\lfloor (im + d^j)h \right\rfloor_n < n \). So we have

\[ \left\{ \left\lfloor (im + d^j)h \right\rfloor_n \mid i \in [0, d-1] \right\} \subseteq \left\{ im + \left\lfloor hd^j \right\rfloor_m \mid i \in [0, d-1] \right\}. \]

By \( \gcd(h, n) = 1 \), we derive that \( \left\{ \left\lfloor (im + d^j)h \right\rfloor_n \mid i \in [0, d-1] \right\} \) have \( d \) distinct elements. Since these two sets both have \( d \) elements, we have

\[ \left\{ \left\lfloor (im + d^j)h \right\rfloor_n \mid i \in [0, d-1] \right\} = \left\{ im + \left\lfloor hd^j \right\rfloor_m \mid i \in [0, d-1] \right\}, \]

and there exists only one element \( i_0 \in [0, d-1] \) such that

\[ 0 < \left\lfloor (i_0m + d^j)h \right\rfloor_n < m. \]

\( \square \)
By lemma 2.1 we rewrite Eq. (3) as

(4) \[ n \parallel T \parallel_{g_i} = \left( \sum_{(i,j) \in B} + \sum_{(i,j) \in C} \right) t_{ij} \left| (im + d^j) h \right|_n. \]

We consider the \( d \) elements of \( A, (i,0) \), where \( i \in [0,d-1] \). By lemma 2.2, we have

\[ \left\{ \left| (im + d^0)h \right|_n \mid i \in [0,d-1] \right\} = \left\{ \mid im + d^0 \mid_m \mid i \in [0,d-1] \right\}. \]

Then for some \( i_0 \in [0,d-1] \), one has \( \mid (i_0m + d^0)h \mid_n = \mid h \mid_m \), so \( (i_0,0) \in B \). For some \( i_1 \in [0,d-1] \), one has \( \mid (i_1m + d^0)h \mid_n = m + \mid h \mid_m \), so \( (i_1,0) \in C \). Then we derive that \( B, C \neq \emptyset \).

Here we set \( |B| = x \) and sort the elements in \( B \) as

\[ B = \{ (\mu_1, \tau_1), (\mu_2, \tau_2), \ldots, (\mu_x, \tau_x) \}, \]

where \( \mu_s, \tau_s \) and \( x \) are integers with \( \mu_s \in [0,d-1] \), \( 0 = \tau_1 \leq \tau_2 \leq \cdots \leq \tau_x \leq k \) and \( x \geq 1 \).

By lemma 2.2, we derive that for any integer \( \tau_s \), there exists at most one element \( \mu_s \in [0,d-1] \) such that \( 0 < \mid (\mu_s + d^s)h \mid_n \). By the enumeration of the elements of \( B \), we know that actually \( 0 = \tau_1 \leq \tau_2 < \cdots < \tau_x \leq k \).

Next we will prove another quality of the sorted elements in \( B \) when \( x \geq 2 \).

**Lemma 2.3.** When \( |B| = x \geq 2 \), for every integer \( a \in [1, x-1] \), we have

\[ m < \left| (\mu_a m + d^a)h \right|_n d^{a+1-\tau_a} < n. \]

**Proof.**

**Case 1.** \( \tau_{a+1} - \tau_a = 1 \).

By the definition of \( B \) we have \( 0 < \left| (\mu_a m + d^a)h \right|_n < m \), thus \( 0 < \left| (\mu_a m + d^a)h \right|_n d < n \). It is clear that \( \left| (\mu_a m + d^a)h \right|_n d \neq m \). Assuming that \( 0 < \left| (\mu_a m + d^a)h \right|_n d < m \), by the definition of \( B \) we also have \( 0 < \left| (\mu_{a+1} m + d^{a+1})h \right|_n < m \). Thus

(5) \[ \left| (\mu_a m + d^a)h \right|_n d - \left| (\mu_{a+1} m + d^{a+1})h \right|_n \in (-m,m). \]

But we have

\[ \left| (\mu_a m + d^a)h \right|_n d - \left| (\mu_{a+1} m + d^{a+1})h \right|_n = \mid -\mu_a m h \mid_n = \mid -\mu_{a+1} m \mid_d m. \]

Since \( \mu_{a+1} \in [1, d-1] \) and \( \gcd(h,n) = 1 \), we have \( \mid -\mu_{a+1} m \mid_d \neq d \). Hence

\[ \mid (\mu_a m + d^a)h \mid_n d - \left| (\mu_{a+1} m + d^{a+1})h \right|_n = ym \] with integer \( y \neq 0 \),

a contradiction to Eq. (5). So that \( m < \left| (\mu_a m + d^a)h \right|_n d < n \).

**Case 2.** \( \tau_{a+1} - \tau_a \geq 2 \).

First, for any integers \( v \in [\tau_a + 1, \tau_a + 1 - 1] \) and \( i \in [1, d-1] \), we have \( (i, v) \in A \) by the definition of \( A \). By definition of \( B \), \( (i, v) \notin B \). By lemma 2.1, we have \( (i, v) \in C \). Then by the definition of \( C \), we have

(6) \[ m < \left| (im + d^v)h \right|_n < n, \]

where \( v \in [\tau_a + 1, \tau_a + 1 - 1] \) and \( i \in [1, d-1] \).
Second, for every \( z \in [0, \tau_{a+1} - \tau_a - 2] \), we will prove that, if \( 0 < |(\mu_a m + d^\tau_a)h| \_n d^z < m \), then \( 0 < |(\mu_a m + d^\tau_a)h| \_n d^{z+1} < m \).

For every \( z \in [0, \tau_{a+1} - \tau_a - 2] \), we let \( v = \tau_a + z + 1 \), and suppose that
\[
0 < |(\mu_a m + d^\tau_a)h| \_n d^z < m.
\]
Then we have
\[
0 < |(\mu_a m + d^\tau_a)h| \_n d^{z+1} < n.
\]
Therefore,
\[
(7) \quad |(\mu_a m + d^\tau_a)h| \_n d^{z+1} = |(\mu_a m + d^\tau_a)hd^{z+1}| \_n = |d^{\tau_a + z + 1}h| \_n = |hd^v| \_n.
\]
By lemma 2.2 we have
\[
(8) \quad \left\{|(im + d^v)h| \_n \mid i \in [0, d - 1]\right\} = \left\{|im + |hd^v| \_m \mid i \in [0, d - 1]\right\}.
\]
Note that \( v = \tau_a + z + 1 \in [\tau_a + 1, \tau_{a+1} - 1] \). By Eq. (6), we have
\[
\left\{|(im + d^v)h| \_n \mid i \in [1, d - 1]\right\} \subset \left\{|im + |hd^v| \_m \mid i \in [1, d - 1]\right\}.
\]
Since these two sets both have \( d - 1 \) elements, we have
\[
(9) \quad \left\{|(im + d^v)h| \_n \mid i \in [1, d - 1]\right\} = \left\{|im + |hd^v| \_m \mid i \in [1, d - 1]\right\}.
\]
Then combining Eq. (8) with Eq. (9), we have
\[
\left\{|(im + d^v)h| \_n \mid i = 0\right\} = \left\{|im + |hd^v| \_m \mid i = 0\right\}.
\]
That is, \( |hd^v| \_n = |hd^v| \_m \). Then by \( 0 < |hd^v| \_m < m \) and Eq. (7), we have
\[
0 < |(\mu_a m + d^\tau_a)h| \_n d^{z+1} < m.
\]

Last, thus we proceed by induction on \( z \in [0, \tau_{a+1} - \tau_a - 2] \). Since \( 0 < |(\mu_a m + d^\tau_a)h| \_n d^z < m \) is true for \( z = 0 \) by the definition of \( B \), we let \( z = \tau_{a+1} - \tau_a - 2 \) and derive that
\[
0 < |(\mu_a m + d^\tau_a)h| \_n d^{\tau_{a+1} - \tau_a - 1} < m
\]
is true. Thus \( 0 < |(\mu_a m + d^\tau_a)h| \_n d^{\tau_{a+1} - \tau_a} < n \). It is clear that \( |(\mu_a m + d^\tau_a)h| \_n d^{\tau_{a+1} - \tau_a} \neq m \).

Assuming that \( 0 < |(\mu_a m + d^\tau_a)h| \_n d^{\tau_{a+1} - \tau_a} < m \), by the definition of \( B \) we also have \( 0 < |(\mu_{a+1} m + d^\tau_{a+1})h| \_n < m \). Thus
\[
(10) \quad |(\mu_a m + d^\tau_a)h| \_n d^{\tau_{a+1} - \tau_a} - |(\mu_{a+1} m + d^\tau_{a+1})h| \_n \in (-m, m).
\]
But we have
\[
|(\mu_a m + d^\tau_a)h| \_n d^{\tau_{a+1} - \tau_a} - |(\mu_{a+1} m + d^\tau_{a+1})h| \_n = |m_{a+1}h| \_d m.
\]
It is a contradiction to Eq. (10). So that \( m < |(\mu_a m + d^\tau_a)h| \_n d^{\tau_{a+1} - \tau_a} < n \). \( \square \)
3. Proof of Theorem 1.4 and Theorem 1.5

Proof of Theorem 1.4. Suppose to the contrary that there exists a subsequence \( T \subset S \) with \( T \neq \emptyset \) and \( \text{ind}(T) \in [1,r] \). We use the same relevant notions defined in last section. Without loss of generality, we assume that \( |B| = x \geq 2 \), because the following proof also holds true by some minor modifications (for example, we view all the \( \sum_{i=1}^{x-1} f(l) \) as 0 when \( x = 1 \)). We could rewrite Eq. (4) as

\[
 n \parallel T \parallel_{g_1} = \sum_{l=1}^{x-1} t_{\mu_l} \tau_l \left| (\mu_l m + d^\tau)h \right|_n + t_{\mu_x} \tau_x \left| (\mu_x m + d^\tau)h \right|_n + \sum_{(i,j) \in C} t_{ij} \left| (im + d^j)h \right|_n.
\]

(11)

For \( l \in [1, x-1] \), we set

\[
t_{\mu_l} \tau_l = s_l d^{\tau_l+1} - \tau_l + t'_{\mu_l} \tau_l,
\]

where \( s_l \geq 0 \) and \( t'_{\mu_l} \tau_l \in [0, d^{\tau_l+1} - \tau_l - 1] \). Then we use three steps to complete the proof.

First, we will prove that \( \sum_{l=1}^{x-1} s_l + \sum_{(i,j) \in C} t_{ij} \leq dr - 1 \). By Eqs. (11) and (12), we have

\[
 n \parallel T \parallel_{g_1} \geq \sum_{l=1}^{x-1} (s_l d^{\tau_l+1} - \tau_l + t'_{\mu_l} \tau_l) \left| (\mu_l m + d^\tau)h \right|_n + t_{\mu_x} \tau_x \left| (\mu_x m + d^\tau)h \right|_n + \sum_{(i,j) \in C} t_{ij} \left| (im + d^j)h \right|_n
\]

(13)

For \( l \in [1, x-1] \), we set

\[
t_{\mu_l} \tau_l = s_l d^{\tau_l+1} - \tau_l + t'_{\mu_l} \tau_l,
\]

where \( s_l \geq 0 \) and \( t'_{\mu_l} \tau_l \in [0, d^{\tau_l+1} - \tau_l - 1] \). Then we use three steps to complete the proof.

First, we will prove that \( \sum_{l=1}^{x-1} s_l + \sum_{(i,j) \in C} t_{ij} \leq dr - 1 \). By Eqs. (11) and (12), we have

\[
 n \parallel T \parallel_{g_1} \geq \sum_{l=1}^{x-1} (s_l d^{\tau_l+1} - \tau_l + t'_{\mu_l} \tau_l) \left| (\mu_l m + d^\tau)h \right|_n + t_{\mu_x} \tau_x \left| (\mu_x m + d^\tau)h \right|_n + \sum_{(i,j) \in C} t_{ij} \left| (im + d^j)h \right|_n
\]

(13)

Hence we have

\[
 n \parallel T \parallel_{g_1} \geq \sum_{l=1}^{x-1} s_l d^{\tau_l+1} - \tau_l \left| (\mu_l m + d^\tau)h \right|_n + \sum_{(i,j) \in C} t_{ij} \left| (im + d^j)h \right|_n
\]

\[
 > \sum_{l=1}^{x-1} s_l m + \sum_{(i,j) \in C} t_{ij} m = \left( \sum_{l=1}^{x-1} s_l + \sum_{(i,j) \in C} t_{ij} \right) m.
\]

We suppose that \( \sum_{l=1}^{x-1} s_l + \sum_{(i,j) \in C} t_{ij} > dr \), and derive

\[
n \parallel T \parallel_{g_1} > \left( \sum_{l=1}^{x-1} s_l + \sum_{(i,j) \in C} t_{ij} \right) m > rn.
\]
Thus \( \text{ind}(T) = \| T \|_{g_1} > r \), a contradiction to \( \text{ind}(T) \in [1, r] \). So we have

\[
\sum_{l=1}^{x-1} s_l + \sum_{(i,j) \in C} t_{ij} \leq dr - 1. \tag{14}
\]

Next, we will prove that \( |n| T \|_{g_1} |_m \neq m \). By Eq. \((13)\), we have

\[
\begin{align*}
|n| T \|_{g_1} |_m & = \left| \sum_{l=1}^{x-1} t_{l}^{t_{\mu_l,t_l}} d^{t_l} h + \sum_{l=1}^{x-1} s_l d^{t_l+1 - t_l} - \sum_{(i,j) \in C} t_{ij} d^j h \right|_m \\
& = \left| h \left( \sum_{l=1}^{x-1} t_{l}^{t_{\mu_l,t_l}} d^{t_l} + \sum_{l=1}^{x-1} s_l d^{t_l+1} + \sum_{(i,j) \in C} t_{ij} d^j \right) \right|_m \\
& = |h(\ast\ast)|_m,
\end{align*}
\]

where

\[
\ast\ast = \sum_{l=1}^{x-1} t_{l}^{t_{\mu_l,t_l}} d^{t_l} + \sum_{l=1}^{x-1} s_l d^{t_l+1} + \sum_{(i,j) \in C} t_{ij} d^j - \sum_{l=1}^{x-1} (d^{t_l+1 - t_l} - 1) d^{t_l} + \sum_{l=1}^{x-1} s_l d^k + \sum_{(i,j) \in C} t_{ij} d^k \\
& \leq -d^{r_1} + d^{r_x} + \left( \sum_{l=1}^{x-1} s_l + \sum_{(i,j) \in C} t_{ij} \right) d^k \\
& \leq -d^{r_1} + d^{r_x} + m - (dr - 1)d^k \leq m - 1. \tag{16}
\]

It is clear that \( \ast\ast > 0 \) by \( T \neq \emptyset \). So we have \( |n| T \|_{g_1} |_m = |h(\ast\ast)|_m \neq m \) by Eqs. \((15)\) and \((17)\).

Last, since \( |n| T \|_{g_1} |_m \neq m \) and \( m \| n \), we have \( |n| T \|_{g_1} |_n \neq n \). Hence \( \text{ind}(T) = \| T \|_{g_1} \) is not an integer and \( T \) is not a zero-sum subsequence of \( S \). It is a contradiction to \( \text{ind}(T) \in [1, r] \). Thus \( S \) is an index-\( r \)-free sequence.

\[\square\]

Proof of Theorem \((7.3)\): Given any fixed integers \( d > 1 \) and \( r \geq 1 \), we take the same \( S \) defined in theorem \((1.4)\) and let \( n > rd^2 \) with \( d \| n \). Then \( S \) is an index-\( r \)-free sequence for any \( k \in \).
by theorem 1.4. Since \( \left\lfloor \frac{m}{d^j} \right\rfloor > \frac{m}{d^j} - 1 \), we calculate the length of \( S \) and have

\[
|S| = \sum_{(i,j) \in A} \left( \frac{m}{d^j} - (dr - 1)d^{k-j} - 1 \right) > \sum_{(i,j) \in [1,d-1] \times [0,k]} \left( \frac{m}{d^j} - (dr - 1)d^{k-j} - 2 \right) + m - (dr - 1)d^k - 1
\]

\[
= (d - 1) \sum_{j \in [0,k]} \left( \frac{m}{d^j} - (dr - 1)d^{k-j} - 2 \right) + m - (dr - 1)d^k - 1
\]

\[
= \left( 1 + \frac{1}{d} - \frac{1}{d^{k+1}} \right) n - (dr - 1)(d^{k+1} + d^k - 1) - 2(k + 1)(d - 1) - 1.
\]

We let \( k = \left\lfloor \frac{1}{2} \ln(n) \right\rfloor > 0 \) and have

\[
|S| > \left( 1 + \frac{1}{d} \right) n + C_1 \sqrt{n} + C_2 \ln(n) + C_3,
\]

where \( C_1, C_2 \) and \( C_3 \) are some constants determined by \( d \) and \( r \). Thus we have proved the theorem. \( \square \)

Therefore, \( t_r(n) \geq n + \frac{n}{d} + O(\sqrt{n}) \) for every sufficiently large integer \( n \) divisible by \( d \), where \( d > 1 \) and \( r \geq 1 \) are constant integers.

4. Concluding Remarks

Given any fixed integers \( d > 1 \) and \( r \geq 1 \). Since \( \left\lfloor \frac{m}{d^j} \right\rfloor \leq \frac{m}{d^j} \), we can also get upper bounds of \( |S| \) in theorem 1.5. Let \( d \) be the least prime factor of \( n \). Generally, \( |S| < n + \frac{n}{d} \). So we have the following conjecture.

**Conjecture 4.1.** Let \( n \) be a composite number, \( C_n \) a cyclic group of order \( n \), and \( d \) the least prime factor of \( n \). Then every sequence \( S \) of length \( |S| = n + \frac{n}{d} \) over \( C_n \) has a zero-sum subsequence \( T \) with \( \text{ind}(T) = 1 \).

**Open Problem.** Determine \( t_r(n) \) for all integers \( n \geq 2 \) and \( r > 0 \).

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