PRODUCT ONE SUBSEQUENCES OVER SUBGROUPS OF A FINITE GROUP

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Abstract. Let $G$ be a finite group, and let $D^{(1)}(G)$ be the smallest integer $t$ such that, every sequence $S$ over $G$ with length $|S| \geq t$ has a nonempty subsequence $T$ over a cyclic subgroup of $G$ with the product of all terms in $T$ in some order equaling to one, the identity element of $G$. In this paper, among other results, we prove that $D^{(1)}(G) \geq |G|$ holds for all finite groups $G$ and characterize all nilpotent finite groups such that the above equality holds. When $G$ is abelian, we also provide a computation formula of $D^{(1)}(G)$ involving Möbius function.

1. Introduction

As in recent papers [10], [14] and [15], we write a finite group $G$ multiplicatively. We say that $S$ is a product-one sequence if its terms can be ordered so that their product equals 1, the identity element of the group.

Let $G$ be a multiplicatively written, finite cyclic group and $g \in G$ with $\text{ord}(g) = |G| = n$. For a sequence $S = (g^{n_1}) \cdot \ldots \cdot (g^{n_l})$ over $G$, where $l \in \mathbb{N}_0$ and $n_1, \ldots, n_l \in [1, n]$, we set

$$\|S\|_g = \frac{n_1 + \ldots + n_l}{n},$$

and then denote by

$$\text{ind}(S) = \min\{\|S\|_h \mid h \in G \text{ with } \text{ord}(h) = n\} \in \mathbb{Q}_{\geq 0}$$

the index of $S$. The index of a sequence is a crucial invariant in the investigation of (minimal) product-one sequences (resp. of product-one free sequences) over cyclic groups. It was first addressed by Lemke and Kleitman ([19]), used as key tool by Geroldinger ([13, page 736]), and then investigated by Gao [7] in a systematical way. Since then it has attracted a great deal of attention from researchers in combinatoric and additive number theory and related areas (see, for example, [7, 11, 20, 21, 27]).
A possible way to generalize the concept of index of sequences from cyclic groups to finite groups is as follows. For any finite (not necessarily abelian) group $G$, we say that a sequence $S$ over $G$ has index 1 if $S$ is a sequence over a cyclic subgroup of $G$ and $\text{ind}(S) = 1$. Let $t(G)$ be the smallest positive integer $\ell$ such that, every sequence $S$ over $G$ with length $|S| \geq \ell$ has a subsequence of index 1.

For any positive integer $n$, let $C_n$ denote the cyclic group of $n$ elements. Lemke and Kleitman made the following conjecture [19].

**Conjecture 1.1.** Let $p$ be a prime. Then $t(C_p) = p$.

In fact, Lemke and Kleitman conjectured that $t(C_n) = n$ for all positive integers $n$, but it was shown recently that $t(C_n) > n$ for infinitely many composite integers $n$ (see [11, 20, 21, 27]). By now we still do not know any good upper bound on $t(G)$. Note also that Conjecture 1.1 is widely open. Thus, to determine $t(G)$ for all finite groups seems to be very difficult. Here we will consider a related problem and determine the invariant $D^{(1)}(G)$, which is defined as the smallest integer $t$, such that every sequence $S$ over $G$ with length $|S| \geq t$ has a product-one subsequence over a cyclic subgroup of $G$.

One reason that we consider here all finite groups (instead of restricting on finite abelian groups) is, in recent years, product-one problems (or zero-sum problems) for non-abelian groups have attracted more and more attention (see, for example, [1, 2, 14, 15, 10, 18]). It has been shown that the Davenport constant $D(G)$ for any finite (not necessarily commutative) group $G$ has some close connection with the Noether number of $G$, an invariant from the algebraic representation theory. In the history, the investigation on product-one problems can be tracked back to 1960's. The celebrated Erdős-Ginzburg-Ziv theorem was first proved for any finite solvable group by Erdős, Ginzburg and Ziv in [3], and then was generalized to any finite group by Olson in [23]. The Davenport constant of any finite group was first investigated by Olson and White in [24].

In this paper, among other results, we will prove the following main results.

**Theorem 1.2.** For every finite group $G$ we have,

$$D^{(1)}(G) \geq |G|.$$ 

**Theorem 1.3.** Let $G$ be a finite nilpotent group. Then, $D^{(1)}(G) = |G|$ if and only if one of the following holds.

1. $G$ is cyclic.
2. $G$ is a $p$-group of exponent $p$, where $p$ is a prime.
3. $G$ is a dihedral 2-group of order at least 8, i.e., $G = D_{2n}$ with $n = 2^s$ for some integer $s \geq 2$. 

Theorem 1.4. Let $G$ be a finite abelian group such that $G = C_{n_1} \times C_{n_2} \times \cdots \times C_{n_r}$ with $1 < n_1 | n_2 | \cdots | n_r$. Then,

$$D^{(1)}(G) = 1 + \sum_{n | n_i} \sum_{d | n_i} \mu(d) \mu(q) \prod_{i=1}^r \left( \frac{n}{d^n(n_i, q)} \right)$$

where $\phi(n)$ is the Euler’s totient function, and $\mu(d)$ is the classical Möbius function.

The rest of this paper is organized as follows. Section 2 provides some notations and concepts which will be used in the sequel. Section 3 deals with $D^{(1)}(G)$ and provides the proofs of Theorem 1.2 and Theorem 1.3. In Section 4 we give a proof for Theorem 1.4. Some related results will be given in the final section.

2. Preliminaries

We adopt the notations and conventions described in detail in [14].

Let $G$ be a finite multiplicative group. The exponent of $G$ is the least common multiple of the orders of all elements of $G$, denote by $\exp(G)$. Denote by $\langle A \rangle$ the subgroup of $G$ generated by $A$, where $A$ is a nonempty subset of $G$. Recall that by a sequence over a group $G$, we mean a finite, unordered sequence where the repetition of elements is allowed. We view sequences over $G$ as elements of the free abelian monoid $F(G)$ and we denote multiplication in $F(G)$ by the bold symbol $\cdot$ rather than by juxtaposition and use brackets for all exponentiation in $F(G)$.

A sequence $S \in F(G)$ can be written in the form $S = g_1 \cdot g_2 \cdot \ldots \cdot g_\ell$, where $|S| = \ell$ is the length of $S$. For $g \in G$, let

- $v_g(S) = |\{i \in [1, \ell] : g_i = g\}|$ denote the multiplicity of $g$ in $S$;

A sequence $T \in F(G)$ is called a subsequence of $S$ and is denoted by $T \mid S$ if $v_g(T) \leq v_g(S)$ for all $g \in G$. Denote by $T^{[-1]} \cdot S$ or $S \cdot T^{[-1]}$ the subsequence of $S$ obtained by removing the terms of $T$ from $S$.

If $S_1, S_2 \in F(G)$, then $S_1 \cdot S_2 \in F(G)$ denotes the sequence satisfying that $v_g(S_1 \cdot S_2) = v_g(S_1) + v_g(S_2)$ for all $g \in G$. For convenience we write

$$g^{[k]} = g \cdot \ldots \cdot g \in F(G)$$

and

$$T^{[k]} = T \cdot \ldots \cdot T \in F(G),$$

for $g \in G$, $T \in F(G)$ and $k \in \mathbb{N}_0$. Let $T^{[−1]} = (T^{[k]})^{[-1]}$.

Suppose $S = g_1 \cdot g_2 \cdot \ldots \cdot g_\ell \in F(G)$. Let

$$\pi(S) = \{g_{\pi(1)} \cdot \ldots \cdot g_{\pi(\ell)} : \tau \text{ a permutation of } [1, \ell]\} \subseteq G$$
denote the set of products of $S$. Let
\[ \Pi(S) = \bigcup_{1 \leq i \leq \ell} \bigcup_{|T|=i} \pi(T) \]
denote the set of all subsequence products of $S$. The sequence $S$ is called
- **squarefree** if $v_{\pi}(S) \leq 1$ for all $g \in G$;
- **product-one** if $1 \in \Pi(S)$;
- **product-one free** if $1 < \Pi(S)$;
- **minimal product-one** if $1 \in \Pi(S)$ and $S$ cannot be factored into two nonempty, product-one subsequences.

Let $B(G)$ be the set of all nonempty product-one sequences over $G$. For any subset $\Omega \subset B(G)$, let $d_\Omega(G)$ be the smallest integer $t$ such that every sequence $S$ over $G$ with length $|S| \geq t$ has a product-one subsequence in $\Omega$. The invariant $d_\Omega(G)$ was first introduced recently in [12] for abelian groups.

Let $r(G)$ be the smallest integer such that $G$ can be generated by $r$ elements. For $\Omega = \bigcup_{H \leq G, r(H) \leq k} B(H)$, let $D^{(k)}(G) = d_\Omega(G)$. Clearly, we have
\[ D^{(1)}(G) \geq D^{(2)}(G) \geq \cdots \geq D^{(r)}(G) = D(G). \]

We need the following well known result ([17, Theorem 5.1.10]).

**Lemma 2.1.** Let $n > 1$ be an integer, and let $S$ be a product-one free sequence over $C_n$ with $|S| = n - 1$. Then, $S = g^{n-1}$ for some generator $g \in C_n$.

### 3. On $D^{(1)}(G) = |G|$

We say that a cyclic subgroup $H$ of $G$ is a **maximal cyclic subgroup** if there is no cyclic subgroup $K$ of $G$ with $H \subsetneq K$. We need the following result.

**Theorem 3.1.** Let $G$ be a finite group, and let $H_1, H_2, \ldots, H_m$ be all distinct maximal cyclic subgroups of $G$. Then,
\[ D^{(1)}(G) = \sum_{i=1}^{m} (|H_i| - 1). \]

Furthermore, if $S$ is a sequence over $G$ with length $|S| = D^{(1)}(G) - 1$ such that $S$ has no nonempty product-one subsequence $T$ with $\langle T \rangle$ being cyclic, then
\[ S = g_1^{\lceil |H_1|-1 \rceil} \cdot \cdots \cdot g_m^{\lceil |H_m|-1 \rceil} \]
where $\langle g_i \rangle = H_i$ for each $i \in [1, m]$. 

Proof. For every $g \in G$, the subgroup $\langle g \rangle$ generated by $g$ is contained in some maximal cyclic subgroup of $G$. It follows that

$$\bigcup_{i=1}^{m} H_i = G.$$ 

Let $S$ be an arbitrary sequence over $G$ of length $|S| \geq 1 + \sum_{i=1}^{m} (|H_i| - 1)$. For every subgroup $H$ of $G$, let $S_H$ denote the subsequence of $S$ consisting of all terms in $H$. Since $\bigcup_{i=1}^{m} H_i = G$, we infer that $\sum_{i=1}^{m} |S_H| \geq |S| \geq 1 + \sum_{i=1}^{m} (|H_i| - 1)$. It follows that $|S| \geq |H_k| = \mathcal{D}(H_k)$ for some $k \in [1, m]$. Hence, $S_k$ has a nonempty product-one subsequence over $H_k$ and so does $S$. This proves that

$$\mathcal{D}^{(1)}(G) \leq 1 + \sum_{i=1}^{m} (|H_i| - 1).$$

To prove $\mathcal{D}^{(1)}(G) \geq 1 + \sum_{i=1}^{m} (|H_i| - 1)$, for every $i \in [1, m]$, take a generator $g_i \in H_i$. Let

$$T = \prod_{i=1}^{m} g_i^{[|H_i|-1]} = \prod_{i=1}^{m} g_i^{[\text{ord}(g_i)-1]}.$$ 

Clearly, $T$ has no nonempty product-one subsequence with its spanning subgroup being cyclic. This proves the above inequality, completing the proof of the first part of the theorem.

Let $S$ be a sequence over $G$ with $|S| = \mathcal{D}^{(1)}(G) - 1 = \sum_{i=1}^{m} (|H_i| - 1)$. Suppose that $S$ has no nonempty product-one subsequence with its spanning subgroup being cyclic. It follows that $S_H$ is product-one free for each $i \in [1, m]$. Therefore,

$$|S_{H_i}| \leq |H_i| - 1 \text{ for each } i \in [1, m].$$

It follows from $\sum_{i=1}^{m} |S_{H_i}| \geq |S| = \sum_{i=1}^{m} (|H_i| - 1)$ that

$$|S_{H_i}| = |H_i| - 1 \text{ for each } i \in [1, m].$$

This together with $S_{H_i}$ being product-one free implies that $S_{H_i} = g_i^{[|H_i|-1]}$ for some generator $g_i$ of $H_i$ by Lemma 2.1, completing the proof.

$\Box$

Remark 3.2. We can simplify the formulation on $\mathcal{D}^{(1)}(G)$ in Theorem 1.4 for some special groups. For the groups listed in Theorem 1.3 we have $\mathcal{D}^{(1)}(G) = |G|$. Let $p$ be a prime, and let $G = C_{p^a} \oplus C_{p^b}$ with $1 \leq a \leq b$. From Theorem 1.4, or Theorem 3.1 we can obtain that $\mathcal{D}^{(1)}(G) = 1 + p^{a-1}(p^{b+1} + p^b + pa - pb - p - a + b - 1)$.

A finite (not necessarily abelian) group $G$ is called cyclic simple if any two maximal cyclic subgroups $H$ and $K$ of $G$ have only the trivial intersection, i.e., $H \cap K = \{1\}$. Our first main result follows from the following theorem.

Theorem 3.3. Let $G$ be a finite group. Then $\mathcal{D}^{(1)}(G) \geq |G|$. Moreover, the equality $\mathcal{D}^{(1)}(G) = |G|$ holds if and only if $G$ is cyclic simple.
Proof. Let $H_1, H_2, \cdots, H_k$ be all distinct maximum cyclic subgroups of $G$. Then
\[ H_1 \cup H_2 \cup \cdots \cup H_k = G. \]

It follows from Theorem 3.1 that
\[
D^{(1)}(G) = 1 + |H_1 \setminus \{1\}| + |H_2 \setminus \{1\}| + \cdots + |H_k \setminus \{1\}| \geq 1 + |H_1 \cup H_2 \cup \cdots \cup H_k \setminus \{1\}|,
\]
so we have
\[
D^{(1)}(G) \geq |G|.
\]
Moreover, the above equality holds if and only if $H_i \cap H_j = \{1\}$ for any two distinct $i, j \in [1, k]$, i.e., if and only if $G$ is cyclic simple.

\[ \square \]

Theorem 3.4. If a finite group $G$ is cyclic simple, then every subgroup $H$ of $G$ is also cyclic simple.

Proof. Assume to the contrary that $H$ is not cyclic simple. By the definition of a cyclic simple group, there exist two distinct maximal cyclic subgroups $H_1$ and $H_2$ of $H$ such that $\{1\} \subsetneq H_1 \cap H_2$. Let $K_1$ and $K_2$ be the maximal cyclic subgroups of $G$ which contain $H_1$ and $H_2$ respectively. Then $\{1\} \subsetneq H_1 \cap H_2 \subsetneq K_1 \cap K_2$. Since $G$ is cyclic simple, we must have $K_1 = K_2 = K$. Therefore, $H_1 \subsetneq K \cap H$ and $H_2 \subsetneq K \cap H$. By the maximality of $H_1$ and $H_2$, we infer that $H_1 = K \cap H = H_2$, a contradiction. Thus $H$ must be cyclic simple, completing the proof.

\[ \square \]

Corollary 3.5. Let $G$ be a finite abelian group. If $G$ is cyclic simple, then either $G$ is cyclic, or $G$ is an elementary abelian $p$-group for some prime $p$.

Proof. Assume to the contrary that $G$ is neither cyclic nor an elementary abelian $p$-group. Then, $G = C_{n_1} \times C_{n_2} \times \cdots \times C_{n_r}$ with $1 < n_1 \mid n_2 \mid \cdots \mid n_r$, $r \geq 2$ and $n_r$ composite. By Theorem 3.4, the subgroup $H = C_{n_1} \times C_{n_2}$ is cyclic simple. Let $x \in C_{n_1}$ with ord($x$) being a prime and let $y \in C_{n_r}$ with ord($y$) = $n_r$. Now the two different cyclic subgroups $\langle y \rangle$ and $\langle xy \rangle$ both have order $n_r$, the maximal value of the order of a cyclic subgroup of $G$. Therefore, both $\langle y \rangle$ and $\langle xy \rangle$ are maximal cyclic subgroups of $G$. But $1 \neq y^{\text{ord}(x)} \in \langle y \rangle \cap \langle xy \rangle$, a contradiction.

\[ \square \]

Corollary 3.6. Let $G$ be a finite group with non-trivial center $Z(G)$, i.e., $|Z(G)| > 1$. If $G$ is cyclic simple and $G$ has some element of composite order, then

1. $G$ has exactly one maximal cyclic subgroup $H$ of composite order.
2. $Z(G) \subset H$.
3. $H$ is a normal subgroup of $G$.
Proof. Let \( H \) be a maximal cyclic subgroup of composite order. Take an arbitrary element \( x \in Z(G) \). Consider the abelian subgroup \( \langle x, H \rangle \) of \( G \), generated by \( x \) and \( H \). Clearly, this subgroup is not an elementary abelian \( p \)-group for any prime \( p \) as \( H \) is a cyclic group of composite order. By Corollary 3.5, \( \langle x, H \rangle \) is cyclic. Hence, \( \langle x, H \rangle = H \) and thus \( \langle x \rangle \subset H \). Therefore, \( Z(G) \subset H \), proving Conclusion 2, and Conclusion 1 follows from the assumption that \( G \) is cyclic simple. It remains to prove \( H \) is normal. Let \( g \in G \), and let \( y \) be a generator of \( H \). Then, \( \text{ord}(gyg^{-1}) = \text{ord}(y) \) is composite. Since \( H \) is the unique maximal cyclic subgroup of \( G \) with composite order \( |H| \), it forces that \( gyg^{-1} \in H \). This proves that \( H \) is normal. \( \square \)

Lemma 3.7. Let \( G \) be a finite non-cyclic \( p \)-group for some prime \( p \). Suppose that \( G \) has exponent larger than \( p \). If \( G \) is cyclic simple, then \( p = 2 \) and \( G \) is the dihedral \( 2 \)-group \( D_{2n} \) with \( n = 2^s \) and \( s \geq 2 \).

Proof. It is well known that \( |Z(G)| > 1 \) as \( G \) is a nontrivial \( p \)-group. Since \( G \) is cyclic simple and has exponent larger than \( p \), by Corollary 3.6 we conclude that \( G \) has exactly one maximal cyclic subgroup \( H \) with order \( |H| > p \), \( G \setminus H \neq \emptyset \) and every element in \( G \setminus H \) is of order \( p \). Let \( a \) be a generator of \( H \) and let \( p^m = \text{ord}(a) = |H| \).

Take an element \( b \in G \setminus H \). Since \( H \) is a normal subgroup of \( G \) by Corollary 3.6, we have \( bab^{-1} \in H \), and thus, \( bab^{-1} = a^k \). Now we have

\begin{equation}
   \tag{3.1}
   b^p = 1, (ba)^p = (ab)^p = 1, \text{ and } ba = a^kb.
\end{equation}

From \( ba = a^kb \), we infer that

\begin{equation}
   \tag{3.2}
   ba^t = a^{tk}b.
\end{equation}

Since \( Z(G) \subset H \) and \( |Z(G)| > 1 \), we obtain that \( a^{p^{m-1}} \in Z(G) \). Therefore, \( ba^{p^{m-1}}b^{-1} = a^{p^{m-1}} \). On the other hand, by \( bab^{-1} = a^k \), we deduce that \( ba^{p^{m-1}}b^{-1} = a^{kp^{m-1}} \). Hence, \( a^{p^{m-1}} = a^{kp^{m-1}} \). This implies that \( p^{m-1} \equiv kp^{m-1} \pmod{p^m} \).

So, equivalently, we have that

\begin{equation}
   \tag{3.3}
   k \equiv 1 \pmod{p}.
\end{equation}

By induction on \( t \geq 2 \) and \( ba^t = a^{tk}b \) we can deduce that

\begin{equation}
   \tag{3.4}
   (ab)^t = a^{1+k+k^2+\cdots+k^{t-1}}b.
\end{equation}
Especially, we have

\[ 1 = (ab)^p = a^{1+k+k^2+\cdots+k^{p-1}}b^p = a^{1+k+k^2+\cdots+k^{p-1}}. \]

This gives that

\[ \frac{k^p - 1}{k - 1} = 1 + k + k^2 + \cdots + k^{p-1} \equiv 0 \pmod{p^m}. \]  

(3.5)

By (3.3) we know that \( k = sp + 1 \) for some integer \( s \). This together with (3.5) gives that

\[ \sum_{i=0}^{p-1} \binom{p}{i}(sp)^{p-i} \equiv 0 \pmod{p^m}. \]

(3.6)

If \( p \geq 3 \), then the left side of (3.6) is equal to \( p^2\alpha + p \not\equiv 0 \pmod{p^m} \) as \( m > 1 \), where \( \alpha = \frac{\sum_{i=0}^{p-2} \binom{p}{i}(sp)^{p-i}}{sp} \) is an integer, giving a contradiction. Thus we must have \( p = 2 \) and \( k = 2s + 1 \equiv -1 \pmod{2^m} \) by (3.6). Therefore,

\[ bab^{-1} = a^{-1}. \]

We show next that

\[ G = \langle a, b \rangle. \]

Assume to the contrary that, \( G \setminus \langle a, b \rangle \neq \emptyset \). Take any \( c \in G \setminus \langle a, b \rangle \). As above, we can prove that

\[ cac^{-1} = a^{-1}. \]

Therefore,

\[ (bc)a(bc)^{-1} = b(cac^{-1})b^{-1} = ba^{-1}b^{-1} = a. \]

So, the subgroup \( \langle bc, a \rangle \) generated by \( bc \) and \( a \) is abelian. By Corollary 3.5 we have that \( \langle bc, a \rangle \) is cyclic. Since \( H \) is a maximal cyclic subgroup of \( G \), we obtain that \( \langle bc, a \rangle = H = \langle a \rangle \). So, \( bc \in \langle bc, a \rangle = H \subset \langle b, a \rangle \) giving a contradiction to the choice of \( c \in G \setminus \langle a, b \rangle \). This proves that \( G = \langle a, b \rangle \), and \( G = D_{2n} \) with \( n = \frac{|G|}{2} = 2^s \) and \( s \geq 2 \). \( \square \)

As a consequence, we obtain the following result.

**Theorem 3.8.** If \( G \) is a finite cyclic simple group, then for every odd prime divisor \( p \) of \( |G| \), each Sylow \( p \)-subgroup of \( G \) is either a \( p \)-group of exponent \( p \) or a cyclic group. Moreover, if \( 2 \mid |G| \), then each Sylow \( 2 \)-subgroup is either an elementary abelian \( 2 \)-group, or a cyclic group, or a dihedral \( 2 \)-group of order at least 8.

We are now ready to provide a proof for the second main result.

**Proof of Theorem 1.3.** If \( G \) is a finite \( p \)-group for some prime \( p \), then the result follows from Lemma 3.7. Now assume that \( |G| \) has at least two distinct
prime divisors. We first assume that the Sylow $p$-subgroup of $G$ is not cyclic for some prime $p \mid |G|$. Let $H$ be the Sylow $p$-subgroup of $G$, and let $K$ be the Sylow $q$-subgroup of $G$ for a prime $q \mid |G|$ with $q \neq p$. Since $G$ is nilpotent, the group $H \times K$ is a subgroup of $G$. It follows from Theorem 3.4 that $HK = H \times K$ is cyclic simple.

Take $x \in K$ with $\text{ord}(x)$ being maximal. Since $H$ is not cyclic, we can take two elements $a, b$ in $H$ with $\langle a \rangle$ and $\langle b \rangle$ are two different maximal cyclic subgroups of $H$. Note that for any $c \in H$ and $z \in K$ we have $cz = zc$ and $\text{ord}(cz) = \text{ord}(c) \text{ord}(z)$. By the maximality of orders of $x, a, b$, we know that both $\langle ax \rangle$ and $\langle bx \rangle$ are maximal cyclic subgroups of $HK = H \times K$. However, $1 \neq x^{\text{ord}(H)} = (ax)^{\text{ord}(H)} = (bx)^{\text{ord}(H)} \in \langle ax \rangle \cap \langle bx \rangle$, yielding a contradiction to $HK$ being cyclic simple. Thus we must have that for every prime $p \mid |G|$, the Sylow $p$-subgroup of $G$ is cyclic.

Thus $G$ is cyclic and we are done. □

4. Proof of Theorem 1.4

Proof of Theorem 1.4. We say an element $g \in G$ is irreducible if the subgroup $\langle g \rangle$ is a maximal cyclic subgroup of $G$. For any positive factor $d$ of $n_r = \exp(G)$, let

$$w(d) = \| \{ g \in G, \text{ord}(g) = d \text{ and } g \text{ is irreducible} \} \|.$$

By Theorem 3.1, we have

$$(4.1) \quad D^{(1)}(G) = 1 + \sum_{d \mid n_r} \frac{w(d)}{\phi(d)} (d - 1).$$

For every positive factor $n$ of $n_r$, let

$$f(n) = \| \{ g \in G, ng = 0 \text{ and } g \text{ is irreducible} \} \|.$$

Then,

$$\sum_{d \mid n} w(d) = f(n).$$

By the Möbius inversion theorem, we obtain that

$$(4.2) \quad w(n) = \sum_{d \mid n} \mu(d) f\left(\frac{n}{d}\right).$$

So, it remains to compute $f(n)$. For every factor $q \mid n_r$, let

$$h(n, q) = \| \{ g \in G, ng = 0, g \in qG \} \|.$$
Let 
\[ n_r = p_1^{u_1} \cdots p_l^{u_l} \]
with \( p_1, \ldots, p_l \) being distinct primes.

By the Inclusion-Exclusion Principle we get
\[
f(n) = h(n, 1) - \sum_{i=1}^{l} h(n, p_i) + \sum_{1 \leq i < j \leq l} h(n, p_ip_j) - \cdots + (-1)^l h(n, p_1p_2 \cdots p_l).
\]

Since \( \mu(d) = 0 \) if \( d \) is not square-free, we obtain that
\[
(4.3) \quad f(n) = \sum_{q|n_r} \mu(q) h(n, q).
\]

Note that
\[
qG = \langle e_1 \rangle \oplus \langle e_2 \rangle \oplus \cdots \oplus \langle e_r \rangle
\]
with \( 1 \leq \frac{n_1}{(n_1, q)} \left| \frac{n_2}{(n_2, q)} \right| \cdots \left| \frac{n_r}{(n_r, q)} \right| \).

Write
\[
qG = \langle e_1 \rangle \oplus \langle e_2 \rangle \oplus \cdots \oplus \langle e_r \rangle
\]
with \( \text{ord}(e_i) = \frac{n_i}{(n_i, q)} \) for every \( i \in [1, r] \). An element \( g = m_1e_1 + m_2e_2 + \cdots + m_re_r \in qG \) satisfies \( ng = 0 \) if and only if
\[
nm_i \equiv 0 \pmod{\frac{n_i}{(n_i, q)}}
\]
for every \( i \in [1, r] \).

Note that the number of solutions for the congruence \( ax \equiv 0 \pmod{v} \) is \( (a, v) \). We infer that \( h(n, q) = \prod_{i=1}^{r} \left( n, \frac{n_i}{(n_i, q)} \right) \). Now the desired result follows from (4.1), (4.2) and (4.3).

\[ \square \]

5. Some Related Results

In this section, we present some related results. Let \( \mathcal{F} \) be a set of some subgroups of a finite group \( G \) and let \( \Omega_{\mathcal{F}} = \cup_{H \in \mathcal{F}} B(H) \). We first recall a result from [12].

**Lemma 5.1.** ([12, Proposition 3.1]) Let \( G \) be a finite group, and let \( \Omega \subset \mathcal{B}(G) \). Then, \( d_{\Omega}(G) < \infty \) if and only if for every element \( g \in G \), \( g^{\text{ord}(g)} \in \Omega \) for some positive integer \( k = k(g) \).

We remark that the above lemma was proved only for the case when \( G \) is abelian in [12]. However, the same proof works for the general case.

The following result regarding \( d_{\Omega_{\mathcal{F}}} \) follows immediately from Lemma 5.1.

**Theorem 5.2.** \( d_{\Omega_{\mathcal{F}}} < \infty \) if and only if \( \cup_{H \in \mathcal{F}} H = G \).
By the definitions of $t(G)$ and $D^{(1)}(G)$, we can easily deduce the following inequality

\begin{equation}
(5.1) \quad t(G) \geq D^{(1)}(G)
\end{equation}

where $G$ is any finite group.

The following proposition presents some special groups for which the equality in (5.1) holds.

**Proposition 5.3.** Let $G$ be a finite group. If $\exp(G) \leq 7$ then $t(G) = D^{(1)}(G)$.

**Proof.** In terms of (5.1), it suffices to prove that $t(G) \leq D^{(1)}(G)$. And this follows from the fact that every minimal product-one sequence over $C_n$ with $n \leq 7$ has index 1 and we are done. \qed

In terms of the proof of Theorem 3.1, we conclude that Conjecture 1.1 is equivalent to the following one.

**Conjecture 5.4.** Let $G$ be a finite $p$-group with $\exp(G) = p$ for some prime $p$. Then, $t(G) = |G| = D^{(1)}(G)$.

We next compute $D^{(2)}(G)$ for a finite elementary abelian 2-group $G$ and we have the following main result.

**Theorem 5.5.** Let $G = C_2^r$ with $r \geq 1$ be an elementary abelian 2-group. Then we have

$$D^{(2)}(G) = 2^{r-1} + 1.$$ 

Let $G$ be a finite abelian group. For each positive integer $k \geq \exp(G)$, let $s_{\leq k}(G)$ be the smallest positive integer $t$ such that every sequence $S$ over $G$ of length $|S| \geq t$ has a nonempty product-one subsequence $T$ with $|T| \leq k$. The invariant $s_{\leq k}(G)$ was studied recently in [22] and [26]. By the definitions of $D^{(k)}(G)$ and $s_{\leq k}(G)$, we can easily obtain the following result.

**Lemma 5.6.** For any finite abelian $G$ and any positive integer $\ell \leq r(G)$, we have

$$D^{(\ell)}(G) \leq s_{\leq \ell+1}(G).$$

**Proof.** Let $S$ be an arbitrary sequence over $G$ with length $|S| = s_{\leq \ell+1}(G)$. By the definition of $s_{\leq \ell+1}(G)$, we obtain that there is a nonempty product-one subsequence $T$ with length $|T| \leq \ell + 1$. Since $T$ is product-one, it follows that $r(\langle T \rangle) \leq |T| - 1 \leq \ell$, completing the proof. \qed
We need the following well known result (see [17, Theorem 5.5.9] for a proof).

**Lemma 5.7.** Let \( G = C_{p^1} \times C_{p^2} \times \cdots \times C_{p^r} \) be a finite abelian \( p \)-group for some prime \( p \). Then, \( D(G) = 1 + \sum_{i=1}^r (p^{e_i} - 1) \).

We now prove the following main lemma.

**Lemma 5.8.** For every positive integer \( r \) and \( \ell \leq r \), we have

\[
D^{(r)}(C_2^r) = s_{\leq \ell+1}(C_2^r).
\]

**Proof.** By Lemma 5.6, it suffices to prove that \( s_{\leq \ell+1}(C_2^r) \leq D^{(r)}(C_2^r) \). Let \( S \) be a sequence over \( C_2^r \) with length \( |S| = D^{(r)}(C_2^r) \). We need to prove that \( S \) has a nonempty product-one subsequence with length not exceeding \( \ell + 1 \). By the definition of \( D^{(r)}(C_2^r) \), \( S \) has a nonempty product-one subsequence \( T \) with \( r(\langle T \rangle) \leq \ell \). By Lemma 5.7 we obtain that \( D(\langle T \rangle) = D(C_2^{r(\langle T \rangle)}) = r(\langle T \rangle) + 1 \), and thus it follows that \( T \) has a nonempty product-one subsequence \( W \) with length \( |W| \leq r(\langle T \rangle) + 1 \leq \ell + 1 \), completing the proof. \( \square \)

**Proof of Theorem 5.5.** By Lemma 5.8, we have that \( D^{(2)}(C_2^r) = s_{\leq 3}(C_2^r) \). Since \( s_{\leq 3}(C_2^r) = 2^{r-1} + 1 \), (which is a known result (see [5, Theorem 7.2] ), we obtain the desired result. \( \square \)

**Remark 5.9.** A subset \( A \) of \( G \) is said to be sum-free if \( A \cap (A + A) = \emptyset \). When \( G = C_2^r \), \( D^{(2)}(G) = s_{\leq 3}(G) \) which is equal to one plus the maximal cardinality of a sum-free subset of \( G \). Sum-free sets have been studied since 1960’s. It was proved in [25] that if \( G = C_p^r \) for some prime \( p = 3k \pm 1 \) then the maximal cardinality of a sum-free set of \( G \) is equal to \( kp^{r-1} \). In particular, when \( p = 2 \), the above result implies that \( D^{(2)}(G) = 2^{r-1} + 1 \), which admits a very direct proof.

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