# ON HALF-FACTORIALITY OF TRANSFER KRULL MONOIDS 

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#### Abstract

Let $H$ be a transfer Krull monoid over a subset $G_{0}$ of an abelian group $G$ with finite exponent. Then every non-unit $a \in H$ can be written as a finite product of atoms, say $a=u_{1} \cdot \ldots \cdot u_{k}$. The set $\mathrm{L}(a)$ of all possible factorization lengths $k$ is called the set of lengths of $a$, and $H$ is said to be half-factorial if $|\mathrm{L}(a)|=1$ for all $a \in H$.

We show that, if $a \in H$ and $\left|\mathrm{L}\left(a^{\lfloor(3 \exp (G)-3) / 2\rfloor}\right)\right|=1$, then the smallest divisor-closed submonoid of $H$ containing $a$ is half-factorial. In addition, we prove that, if $G_{0}$ is finite and $\left|\mathrm{L}\left(\prod_{g \in G_{0}} g^{2 \operatorname{ord}(g)}\right)\right|=1$, then $H$ is half-factorial.


## 1. Introduction

Let $H$ be a monoid. If an element $a \in H$ has a factorization $a=u_{1} \cdot \ldots \cdot u_{k}$, where $k \in \mathbb{N}$ and $u_{1}, \ldots, u_{k}$ are atoms of $H$, then $k$ is called a factorization length of $a$, and the set $\mathrm{L}(a)$ of all possible $k$ is referred to as the set of lengths of $a$. The monoid $H$ is said to be half-factorial (half-factorial) if $|\mathrm{L}(a)|=1$ for every $a \in H$. Half-factoriality has been a central topic in factorization theory since the early days of this field (e.g., see $[3,4,15,19,5,11,16])$.

Given $a \in H$, let $\llbracket a \rrbracket=\{b \in H \mid b$ divides some power of $a\}$ be the smallest divisorclosed submonoid of $H$ containing $a$. Then $\llbracket a \rrbracket$ is half-factorial if and only if $\left|\mathrm{L}\left(a^{n}\right)\right|=1$ for all $n \in \mathbb{N}$, and $H$ is half-factorial if and only if $\llbracket c \rrbracket$ is half-factorial for every $c \in H$. It is thus natural to ask:

Does there exist an integer $N \in \mathbb{N}$ such that, if $a \in H$ and $\left|\mathrm{L}\left(a^{N}\right)\right|=1$, then $\llbracket a \rrbracket$ is half-factorial? (Note that, if $\llbracket a \rrbracket$ is half-factorial for some $a \in H$, then of course $\left|\mathrm{L}\left(a^{k}\right)\right|=1$ for every $k \in \mathbb{N}$.)
We answer this question affirmatively for transfer Krull monoids over finite abelian groups, and we study the smallest $N$ having the above property (Theorems 1.1 and 1.2).

Transfer Krull monoids and transfer Krull domains are a recently introduced class of monoids and domains including, among others, all commutative Krull domains and wide classes of non-commutative Dedekind domains (see Section 2 and [8] for a survey).

Let $H$ be a transfer Krull monoid over a subset $G_{0}$ of an abelian group $G$. Then $H$ is half-factorial if and only if the monoid $\mathcal{B}\left(G_{0}\right)$ of zero-sum sequences over $G_{0}$ is half-factorial (in this case, we also say that the set $G_{0}$ is half-factorial). It is a standing conjecture that

2010 Mathematics Subject Classification. 11B30, 11R27, 13A05, 13F05, 20M13.
Key words and phrases. Transfer Krull monoids, zero-sum sequences, sets of lengths, half-factorial. The last-named author was supported by the Austrian Science Fund (FWF Project P28864-N35).
every abelian group has a half-factorial generating set, which implies that every abelian group can be realized as the class group of a half-factorial Dedekind domain ([9]).

Suppose now that $H$ is a commutative Krull monoid with class group $G$ and that every class contains a prime divisor. It is a classic result that $H$ is half-factorial if and only if $|G| \leq 2$, and it turns out that, also for $|G| \geq 3$, half-factorial subsets (and minimal non-half-factorial subsets) of the class group $G$ play a crucial role in a variety of arithmetical questions (see [10, Chapter 6.7], [14]). However, we are far away from a good understanding of half-factorial sets in finite abelian groups (see [21] for a survey, and [17, 18, 22]). To mention one open question, the maximal size of half-factorial subsets is unknown even for finite cyclic groups ([18]). Our results open the door to a computational approach to the study of half-factorial sets.

More in detail, denote by $\operatorname{hf}(H)$ the infimum of all $N \in \mathbb{N}$ with the following property:

$$
\text { If } a \in H \text { and }\left|\mathrm{L}\left(a^{N}\right)\right|=1 \text {, then } \llbracket a \rrbracket \text { is half-factorial. }
$$

(Here, as usual, we assume $\inf \emptyset=\infty$.) We call $h f(H)$ the half-factoriality index of $H$. If $H$ is not half-factorial, then $\operatorname{hf}(H)$ is the infimum of all $N \in \mathbb{N}$ with the property that $\left|\mathrm{L}\left(a^{N}\right)\right| \geq 2$ for every $a \in H$ such that $\llbracket a \rrbracket$ is not half-factorial.

Theorem 1.1. Let $H$ be a transfer Krull monoid over a finite subset $G_{0}$ of an abelian group $G$ with finite exponent. The following are equivalent:
(a) $H$ is half-factorial.
(b) $\mathrm{hf}(H)=1$.
(c) $G_{0}$ is half-factorial.
(d) $\left|\mathrm{L}\left(\prod_{g \in G_{0}} g^{2 \operatorname{ord}(g)}\right)\right|=1$.

We observe that in general if $H$ is half-factorial, then $\operatorname{hf}(H)=1$. But if $H$ is a transfer Krull monoid over a subset of a torsion free group, then $\operatorname{hf}(H)=1$ does not imply that $H$ is half-factorial (see Example 2.4.1). Furthermore, for every $n \in \mathbb{N}$, there exists a Krull monoid $H$ with finite class group such that $\mathrm{hf}(H)=n$ (see Example 2.4.2).

Theorem 1.2. Let $H$ be a transfer Krull monoid over an abelian group $G$.

1. $\operatorname{hf}(H)<\infty$ if and only if $\exp (G)<\infty$.
2. If $\exp (G)<\infty$ and $|G| \geq 3$, then $\exp (G) \leq h f(H) \leq \frac{3}{2}(\exp (G)-1)$.
3. If $G$ is cyclic or $\exp (G) \leq 6$, then $\operatorname{hf}(H)=\exp (G)$.

We postpone the proofs of Theorems 1.1 and 1.2 to Section 3.

## 2. Preliminaries

Our notation and terminology are consistent with [10]. Let $\mathbb{N}$ be the set of positive integers, let $\mathbb{N}_{0}=\mathbb{N} \cup\{0\}$, and let $\mathbb{Q}$ be the set of rational numbers. For integers $a, b \in \mathbb{Z}$, we denote by $[a, b]=\{x \in \mathbb{Z} \mid a \leq x \leq b\}$ the discrete, finite interval between $a$ and $b$.
Atomic monoids. By a monoid, we mean an associative semigroup with identity, and if not stated otherwise we use multiplicative notation. Let $H$ be a monoid with identity $1=1_{H} \in H$. An element $a \in H$ is said to be invertible (or a unit) if there exists an element
$a^{\prime} \in H$ such that $a a^{\prime}=a^{\prime} a=1$. The set of invertible elements of $H$ will be denoted by $H^{\times}$, and we say that $H$ is reduced if $H^{\times}=\{1\}$. The monoid $H$ is said to be unit-cancellative if for any two elements $a, u \in H$, each of the equations $a u=a$ or $u a=a$ implies that $u \in H^{\times}$. Clearly, every cancellative monoid is unit-cancellative.
Suppose that $H$ is unit-cancellative. An element $u \in H$ is said to be irreducible (or an atom) if $u \notin H^{\times}$and for any two elements $a, b \in H, u=a b$ implies that $a \in H^{\times}$or $b \in H^{\times}$. Let $\mathcal{A}(H)$ denote the set of atoms of $H$. We say that $H$ is atomic if every non-unit is a finite product of atoms. If $H$ satisfies the ascending chain condition on principal left ideals and on principal right ideals, then $H$ is atomic ([7, Theorem 2.6]). If $a \in H \backslash H^{\times}$and $a=u_{1} \ldots u_{k}$, where $k \in \mathbb{N}$ and $u_{1}, \ldots, u_{k} \in \mathcal{A}(H)$, then $k$ is a factorization length of $a$, and

$$
\mathrm{L}_{H}(a)=\mathrm{L}(a)=\{k \in \mathbb{N} \mid k \text { is a factorization length of } a\}
$$

denotes the set of lengths of $a$. It is convenient to set $\mathrm{L}(a)=\{0\}$ for all $a \in H^{\times}$.
A transfer Krull mononid is a monoid $H$ having a weak transfer homomorphism (in the sense of [2, Definition 2.1]) $\theta: H \rightarrow \mathcal{B}\left(G_{0}\right)$, where $\mathcal{B}\left(G_{0}\right)$ is the monoid of zero-sum sequences over a subset $G_{0}$ of an abelian group $G$. If $H$ is a commutative Krull monoid with class group $G$ and $G_{0} \subset G$ is the set of classes containing prime divisors, then there is a weak transfer homomorphism $\theta: H \rightarrow \mathcal{B}\left(G_{0}\right)$. Beyond that, there are wide classes of non-commutative Dedekind domains having a weak transfer homomorphism to a monoid of zero-sum sequences ([25, Theorem 1.1], [24, Theorem 4.4]). We refer to [8, 13] for surveys on transfer Krull monoids. If $\theta: H \rightarrow \mathcal{B}\left(G_{0}\right)$ is a weak transfer homomorphism, then sets of lengths in $H$ and in $\mathcal{B}\left(G_{0}\right)$ coincide ([2, Lemma 2.7]) and thus the statements of Theorems 1.1 and 1.2 can be proved in the setting of monoids of zero-sum sequences.

Monoids of zero-sum sequences. Let $G$ be an abelian group and let $G_{0} \subset G$ be a non-empty subset. Then $\left\langle G_{0}\right\rangle$ denotes the subgroup generated by $G_{0}$. In Additive Combinatorics, a sequence (over $G_{0}$ ) means a finite unordered sequence of terms from $G_{0}$ where repetition is allowed, and (as usual) we consider sequences as elements of the free abelian monoid with basis $G_{0}$. Let

$$
S=g_{1} \cdot \ldots \cdot g_{\ell}=\prod_{g \in G_{0}} g^{\mathrm{v}_{g}(S)} \in \mathcal{F}\left(G_{0}\right)
$$

be a sequence over $G_{0}$. We call

$$
\begin{aligned}
\operatorname{supp}(S) & =\left\{g \in G \mid \mathrm{v}_{g}(S)>0\right\} \subset G \text { the support of } S, \\
|S| & =\ell=\sum_{g \in G} \mathrm{v}_{g}(S) \in \mathbb{N}_{0} \text { the length of } S, \\
\sigma(S) & =\sum_{i=1}^{\ell} g_{i} \text { the sum of } S, \\
\text { and } \Sigma(S) & =\left\{\sum_{i \in I} g_{i} \mid \emptyset \neq I \subset[1, \ell]\right\} \text { the set of subsequence sums of } S .
\end{aligned}
$$

The sequence $S$ is said to be

- zero-sum free if $0 \notin \Sigma(S)$,
- a zero-sum sequence if $\sigma(S)=0$,
- a minimal zero-sum sequence if it is a nontrivial zero-sum sequence and every proper subsequence is zero-sum free.
The set of zero-sum sequences $\mathcal{B}\left(G_{0}\right)=\left\{S \in \mathcal{F}\left(G_{0}\right) \mid \sigma(S)=0\right\} \subset \mathcal{F}\left(G_{0}\right)$ is a submonoid, and the set of minimal zero-sum sequences is the set of atoms of $\mathcal{B}\left(G_{0}\right)$. For any arithmetical invariant $*(H)$ defined for a monoid $H$, we write $*\left(G_{0}\right)$ instead of $*\left(\mathcal{B}\left(G_{0}\right)\right)$. In particular, $\mathcal{A}\left(G_{0}\right)=\mathcal{A}\left(\mathcal{B}\left(G_{0}\right)\right)$ is the set of atoms of $\mathcal{B}\left(G_{0}\right)$ and $\operatorname{hf}\left(G_{0}\right)=\operatorname{hf}\left(\mathcal{B}\left(G_{0}\right)\right)$.

Let $G$ be an abelian group. We denote by $\exp (G)$ the exponent of $G$ which is the least common multiple of the orders of all elements of $G$. Let $r \in \mathbb{N}$ and let $\left(e_{1}, \ldots, e_{r}\right)$ be an $r$-tuple of elements of $G$. Then $\left(e_{1}, \ldots, e_{r}\right)$ is said to be independent if $e_{i} \neq 0$ for all $i \in[1, r]$ and if for all $\left(m_{1}, \ldots, m_{r}\right) \in \mathbb{Z}^{r}$ an equation $m_{1} e_{1}+\ldots+m_{r} e_{r}=0$ implies that $m_{i} e_{i}=0$ for all $i \in[1, r]$. Suppose $G$ is finite. The $r$-tuple $\left(e_{1}, \ldots, e_{r}\right)$ is said to be a basis of $G$ if it is independent and $G=\left\langle e_{1}\right\rangle \oplus \ldots \oplus\left\langle e_{r}\right\rangle$. For every $n \in \mathbb{N}$, we denote by $C_{n}$ an additive cyclic group of order $n$. Since $G \cong C_{n_{1}} \oplus \ldots \oplus C_{n_{r}}, r=\mathrm{r}(G)$ is the rank of $G$ and $n_{r}=\exp (G)$ is the exponent of $G$.

Let $G_{0} \subset G$ be a non-empty subset. For a sequence $S=g_{1} \cdot \ldots \cdot g_{\ell} \in \mathcal{F}\left(G_{0}\right)$, we call

$$
\begin{aligned}
\mathrm{k}(S) & =\sum_{i=1}^{l} \frac{1}{\operatorname{ord}\left(g_{i}\right)} \in \mathbb{Q}_{\geq 0} \quad \text { the cross number of } S, \text { and } \\
\mathrm{K}\left(G_{0}\right) & =\max \left\{\mathrm{k}(S) \mid S \in \mathcal{A}\left(G_{0}\right)\right\} \quad \text { the cross number of } G_{0}
\end{aligned}
$$

For the relevance of cross numbers in the theory of non-unique factorizations, see [18, 20, 23] and [10, Chapter 6].

The set $G_{0}$ is called

- half-factorial if the monoid $\mathcal{B}\left(G_{0}\right)$ is half-factorial;
- non-half-factorial if the monoid $\mathcal{B}\left(G_{0}\right)$ is not half-factorial;
- minimal non-half-factorial if $G_{0}$ is not half-factorial but all its proper subsets are;
- an LCN-set if $\mathrm{k}(A) \geq 1$ for all atoms $A \in \mathcal{A}\left(G_{0}\right)$.

The following simple result ([10, Proposition 6.7.3]) will be used throughout the paper without further mention.

Lemma 2.1. Let $G$ be a finite abelian group and $G_{0} \subset G$ a subset. Then the following statements are equivalent:
(a) $G_{0}$ is half-factorial.
(b) $\mathrm{k}(U)=1$ for every $U \in \mathcal{A}\left(G_{0}\right)$.
(c) $\mathrm{L}(B)=\{\mathrm{k}(B)\}$ for every $B \in \mathcal{B}\left(G_{0}\right)$.

Lemma 2.2. Let $G$ be a finite group, let $G_{0} \subset G$ be a subset, let $S$ be a zero-sum sequence over $G_{0}$, and let $A$ be a minimal zero-sum sequence over $G_{0}$.

1. If $\mathrm{k}(A) \neq 1$, then $\left|\mathrm{L}\left(A^{\exp (G)}\right)\right| \geq 2$.
2. If there exists a zero-sum subsequence $T$ of $S$ such that $|\mathrm{L}(T)| \geq 2$, then $|\mathrm{L}(S)| \geq 2$.
3. If $\mathrm{k}(A)<1$ and $\mathrm{k}(A)$ is minimal over all minimal zero-sum sequences over $G_{0}$, then

$$
\left|\mathrm{L}\left(A^{\left\lceil\frac{\operatorname{ord}(g)}{\operatorname{vg}(A)}\right\rceil}\right)\right| \geq 2, \quad \text { for all } g \in \operatorname{supp}(A)
$$

Proof. 1. Suppose $\mathrm{k}(A) \neq 1$ and let $A=g_{1} \cdot \ldots \cdot g_{\ell}$, where $\ell \in \mathbb{N}$ and $g_{1}, \ldots, g_{\ell} \in G_{0}$. Then

$$
A^{\exp (G)}=\left(g_{1}^{\operatorname{ord}\left(g_{1}\right)}\right)^{\frac{\exp (G)}{\operatorname{ord}\left(g_{1}\right)}} \cdot \ldots \cdot\left(g_{\ell}^{\operatorname{ord}\left(g_{\ell}\right)}\right)^{\frac{\exp (G)}{\operatorname{ord}\left(g_{\ell}\right)}}
$$

which implies that

$$
\left\{\exp (G), \sum_{i=1}^{\ell} \frac{\exp (G)}{\operatorname{ord}\left(g_{i}\right)}\right\}=\{\exp (G), \exp (G) \mathrm{k}(A)\} \subset \mathrm{L}\left(A^{\exp (G)}\right)
$$

It follows by $\mathrm{k}(A) \neq 1$ that $\left|\mathrm{L}\left(A^{\exp (G)}\right)\right| \geq 2$.
2. Suppose $T$ is a zero-sum subsequence of $S$ with $|\mathrm{L}(T)| \geq 2$. It follows by $\mathrm{L}(S) \supset$ $\mathrm{L}(T)+\mathrm{L}\left(S T^{-1}\right)$ that $|\mathrm{L}(S)| \geq|\mathrm{L}(T)| \geq 2$.
3. Suppose $\mathrm{k}(A)<1$ and $\mathrm{k}(A)$ is minimal over all minimal zero-sum sequences over $G_{0}$. Let $g \in \operatorname{supp}(A)$. Then there exist $s \in \mathbb{N}$ and minimal zero-sum sequences $W_{1}, \ldots, W_{s}$ such that

$$
A^{\left[\frac{\operatorname{ord}(g)}{\mathrm{v}_{g}(A)}\right\rceil}=g^{\operatorname{ord}(g)} \cdot W_{1} \cdot \ldots \cdot W_{s} .
$$

Since

$$
\mathrm{k}\left(A^{\left\lceil\frac{\operatorname{ord}(g)}{\mathrm{v}_{g}(A)}\right\rceil}\right)=\left\lceil\frac{\operatorname{ord}(g)}{\mathrm{v}_{g}(A)}\right\rceil \mathrm{k}(A)=1+\sum_{i=1}^{s} \mathrm{k}\left(W_{i}\right)>(1+s) \mathrm{k}(A)
$$

we have $\left\lceil\frac{\operatorname{ord}(g)}{\mathrm{V}_{g}(A)}\right\rceil \neq s+1$ and hence $\left|\mathrm{L}\left(A^{\left\lceil\frac{\operatorname{ord}(g)}{\mathrm{v}_{g}(A)}\right\rceil}\right)\right| \geq 2$.
For commutative and finitely generated monoids, we have the following result.
Proposition 2.3. Let $H$ be a commutative unit-cancellative monoid. If $H_{\mathrm{red}}$ is finitely generated, then $\operatorname{hf}(H)$ is finite.
Proof. We may assume that $H$ is reduced and not half-factorial. Suppose $H$ is finitely generated and suppose $\mathcal{A}(H)=\left\{u_{1}, \ldots, u_{n}\right\}$, where $n \in \mathbb{N}$. Set $A_{0}=\left\{\prod_{i \in I} u_{i} \mid \emptyset \neq I \subset\right.$ $[1, n]\}$. Then $A_{0}$ is finite and hence there exists $M \in \mathbb{N}$ such that $\left|\mathrm{L}\left(a_{0}^{M}\right)\right| \geq 2$ for all $a_{0} \in A_{0}$ with $\llbracket a_{0} \rrbracket$ not half-factorial. Let $a \in H \backslash H^{\times}$such that $\llbracket a \rrbracket$ is not half-factorial. It suffices to show that $\left|\mathrm{L}\left(a^{M}\right)\right| \geq 2$. Suppose $a=u_{1}^{k_{1}} \cdot \ldots \cdot u_{n}^{k_{n}}$, where $k_{1}, \ldots, k_{n} \in \mathbb{N}_{0}$. Set $I_{0}=\left\{i \in[1, n] \mid k_{i} \geq 1\right\}$ and $a_{0}=\prod_{i \in I} u_{i}$. Then $a_{0}$ divides $a$ and $\llbracket a_{0} \rrbracket=\llbracket a \rrbracket$ is not half-factorial, whence $\left|\mathrm{L}\left(a_{0}^{M}\right)\right| \geq 2$ and $\left|\mathrm{L}\left(a^{M}\right)\right| \geq 2$.

If $G_{0}$ is a finite subset of an abelian group, then $\mathcal{B}\left(G_{0}\right)$ is finitely generated ( $[10$, Theorem 3.4.2]) and thus $\operatorname{hf}\left(G_{0}\right)<\infty$. We refer to [6, Sections 3.2 and 3.3] and [12] for semigroups of ideals and semigroups of modules that are finitely generated unit-cancellative but not necessarily cancellative.
Examples 2.4. The following examples will help up to illustrate some important points.

1. Let $\left(e_{1}, e_{2}\right)$ be a basis of $\mathbb{Z}^{2}$ and let $G_{0}=\left\{e_{1},-e_{1}, e_{2},-e_{2}, e_{1}+e_{2},-e_{1}-e_{2}\right\}$. Then $\mathcal{A}\left(G_{0}\right)=\left\{e_{1}\left(-e_{1}\right), e_{2}\left(-e_{2}\right),\left(e_{1}+e_{2}\right)\left(-e_{1}-e_{2}\right), e_{1} e_{2}\left(-e_{1}-e_{2}\right),\left(-e_{1}\right)\left(-e_{2}\right)\left(e_{1}+e_{2}\right)\right\}$. Since $e_{1}\left(-e_{1}\right) \cdot e_{2}\left(-e_{2}\right) \cdot\left(e_{1}+e_{2}\right)\left(-e_{1}-e_{2}\right)=e_{1} e_{2}\left(-e_{1}-e_{2}\right) \cdot\left(-e_{1}\right)\left(-e_{2}\right)\left(e_{1}+e_{2}\right)$, we obtain $G_{0}$ is not half-factorial. Furthermore, we have $G_{1}$ is half-factorial for every nonempty proper subset $G_{1} \subsetneq G_{0}$. Let $A \in \mathcal{B}\left(G_{0}\right)$. If $\operatorname{supp}(A)=G_{0}$, then $|\mathrm{L}(A)| \geq 2$ and $\llbracket A \rrbracket=\mathcal{B}\left(G_{0}\right)$ is not half-factorial. If $\operatorname{supp}(A) \subsetneq G_{0}$, then $\llbracket A \rrbracket=\mathcal{B}(\operatorname{supp}(A))$ is half-factorial and $|\mathrm{L}(A)|=1$. Therefore $\operatorname{hf}\left(G_{0}\right)=1$.
2. Let $G$ be a cyclic group with order $n$ and let $g \in G$ with $\operatorname{ord}(g)=n$, where $n \in \mathbb{N}_{\geq 3}$. Set $G_{0}=\{g,-g\}$. Then $G_{0}$ is not half-factorial. Let $A_{0}=g(-g)$. Then $\llbracket A_{0} \rrbracket$ is not half-factorial and $\left|\mathrm{L}\left(A_{0}^{n-1}\right)\right|=1$, whence $\operatorname{hf}\left(G_{0}\right) \geq n$. Let $A \in \mathcal{B}\left(G_{0}\right)$ with $\llbracket A \rrbracket$ is not half-factorial. Then $\operatorname{supp}(A)=G_{0}$ and $A_{0}$ divides $A$, whence $\left|\mathrm{L}\left(A^{n}\right)\right| \geq 2$. Therefore $\operatorname{hf}\left(G_{0}\right)=n$. Let $G \cong C_{2}^{2}$ and let $\left(e_{1}, e_{2}\right)$ be a basis of $G$. Set $G_{1}=\left\{e_{1}, e_{2}, e_{1}+e_{2}\right\}$. Then $G_{1}$ is not half-factorial. Let $A_{1}=e_{1} e_{2}\left(e_{1}+e_{2}\right)$. Then $\llbracket A_{1} \rrbracket$ is not half-factorial and $\left|\mathrm{L}\left(A_{1}\right)\right|=1$, whence $\operatorname{hf}\left(G_{1}\right) \geq 2$. Let $A \in \mathcal{B}\left(G_{1}\right)$ with $\llbracket A \rrbracket$ is not half-factorial. Then $\operatorname{supp}(A)=G_{1}$ and $A_{1}$ divides $A$, whence $\left|\mathrm{L}\left(A^{2}\right)\right| \geq 2$. Therefore $\mathrm{hf}\left(G_{1}\right)=2$.
3. Let $H$ be a bifurcus moniod (i.e. $2 \in \mathrm{~L}(a)$ for all $\left.a \in H \backslash\left(H^{\times} \cup \mathcal{A}(H)\right)\right)$. For examples, see [1, Examples 2.1 and 2.2]. Since for every $a \in H \backslash H^{\times}$, we have $\{2,3\} \subset \mathrm{L}\left(a^{3}\right)$, it follows that $\mathrm{hf}(H) \leq 3$ and $\mathrm{hf}(H)$ is the minimal integer $t \in \mathbb{N}$ such that $\left|\mathrm{L}\left(a^{t}\right)\right| \geq 2$ for all $a \in H \backslash H^{\times}$. Therefore $\operatorname{hf}(H)=3$ if and only if there exists $a_{0} \in \mathcal{A}(H)$ such that $\mathrm{L}\left(a_{0}^{2}\right)=\{2\}$.
4. Let $H \subset F=F^{\times} \times\left[p_{1}, \ldots, p_{s}\right]$ be a non-half factorial finitely primary monoid of rank $s$ and exponent $\alpha$ (see [10, Definition 2.9.1]). For every $a=\epsilon p_{1}^{t_{1}} \ldots p_{s}^{t_{s}} \in F$, we define $\|a\|=t_{1}+\ldots+t_{s}$, where $t_{1}, \ldots, t_{s} \in \mathbb{N}_{0}$ and $\epsilon \in F^{\times}$. Let $a \in H \backslash H^{\times}$. Since $H$ is primary, we have $H=\llbracket a \rrbracket$ is not half-factorial. Thus $\operatorname{hf}(H)$ is the minimal integer $t \in \mathbb{N}$ such that $\left|\mathrm{L}\left(a^{t}\right)\right| \geq 2$ for all $a \in H \backslash H^{\times}$. Suppose $a_{0} \in H$ with $\left\|a_{0}\right\|=\min \left\{\|a\|: a \in H \backslash H^{\times}\right\}$. Then $a_{0} \in \mathcal{A}(H)$ and $\mathrm{L}\left(a_{0}^{2}\right)=\{2\}$, whence $\operatorname{hf}(H) \geq 3$.

If $H \backslash H^{\times}=\left(p_{1} \ldots p_{s}\right)^{\alpha} F$ and $s \geq 2$, then $H$ is bifurcus and hence $\operatorname{hf}(H)=3$. Suppose $s=1$ and $H \backslash H^{\times}=\left(p_{1}\right)^{\alpha} F$. Let $b=\epsilon p^{\beta} \in H$. Then $p^{3 \alpha}$ divides $b^{4}$. It follows by $p^{3 \alpha}=\left(p^{\alpha}\right)^{3}=p^{\alpha+1} p^{2 \alpha-1}$ that $\left|\mathrm{L}\left(b^{4}\right)\right| \geq 2$, whence $\mathrm{hf}(H) \leq 4$. If $3 \beta \geq 4 \alpha$, then $p^{3 \alpha}$ divides $b^{3}$ and hence $\left|\mathrm{L}\left(b^{3}\right)\right| \geq 2$. If $3 \beta \leq 4 \alpha-2$, then $b$ is an atom and $b^{3}=\epsilon^{3} p^{2 \alpha-1} p^{3 \beta-(2 \alpha-1)}$, whence $\left|\mathrm{L}\left(b^{3}\right)\right| \geq 2$. If $3 \beta=4 \alpha-1$, then $\mathrm{L}\left(b^{3}\right)=\{3\}$. Put all together, if $\alpha \equiv 1 \bmod 3$, then $\operatorname{hf}(H)=4$. Otherwise $\operatorname{hf}(H)=3$.

## 3. Proof of main theorem

Proposition 3.1. Let $G_{0} \subset G$ be a non half-factorial subset and let $S$ be a zero-sum sequence over $G_{0}$ with $\operatorname{supp}(S)=G_{0}$.

1. If $G_{0}$ is an $L C N$-set, then $\left|\mathrm{L}\left(\prod_{g \in G_{0}} g^{\operatorname{ord}(g)}\right)\right| \geq 2$.
2. If $\left|G_{0}\right|=2$, then $\left|\mathrm{L}\left(\prod_{g \in G_{0}} g^{\operatorname{ord}(g)}\right)\right| \geq 2$.
3. If $G_{0}$ is a minimal non half-factorial subset, then $\left|\mathrm{L}\left(S^{\exp (G)}\right)\right| \geq 2$.
4. If $\left|\left\{g \in G_{0} \mid \operatorname{ord}(g) / \mathrm{v}_{g}(S)=\exp (G)\right\}\right| \leq 1$, then $\left|\mathrm{L}\left(S^{\exp (G)}\right)\right| \geq 2$.

Proof. 1. Suppose $G_{0}$ is an LCN-set. Since $G_{0}$ is not half-factorial, there exists a minimal zero-sum sequence $T$ over $G_{0}$ such that $\mathrm{k}(T)>1$. Note that $T$ is a subsequence of $\prod_{g \in G_{0}} g^{\operatorname{ord}(g)}$. Then there exits $W_{1}, \ldots, W_{l} \in \mathcal{A}\left(G_{0}\right)$ such that

$$
\prod_{g \in G_{0}} g^{\operatorname{ord}(g)}=T \cdot W_{1} \cdot \ldots \cdot W_{l}
$$

Thus $\mathrm{k}\left(\prod_{g \in G_{0}} g^{\operatorname{ord}(g)}\right)=\left|G_{0}\right|=\mathrm{k}(T)+\sum_{i=1}^{l} \mathrm{k}\left(W_{i}\right)>1+l$. The assertion follows by $\left\{\left|G_{0}\right|, 1+l\right\} \subset \mathrm{L}\left(\prod_{g \in G_{0}} g^{\operatorname{ord}(g)}\right)$.
2. Suppose $\left|G_{0}\right|=2$ and let $G_{0}=\left\{g_{1}, g_{2}\right\}$. If $G_{0}$ is an LCN-set, the assertion follows by 1 .. Suppose there exists a minimal zero-sum sequence $T$ over $G_{0}$ with $\mathrm{k}(T)<1$. Let $T_{0}=g_{1}^{l_{1}} \cdot g_{2}^{l_{2}}$ be the minimal zero-sum sequence over $G_{0}$ such that $\mathrm{k}\left(T_{0}\right)$ is minimal. If $\min \left\{\frac{\operatorname{ord}\left(g_{1}\right)}{l_{1}}, \frac{\operatorname{ord}\left(g_{2}\right)}{l_{2}}\right\} \leq 2$, say $\frac{\operatorname{ord}\left(g_{1}\right)}{l_{1}} \leq 2$ then

$$
T_{0}^{2}=g_{1}^{\operatorname{ord}\left(g_{1}\right)} \cdot W, \text { where } W \text { is non-empty zero-sum sequence }
$$

Thus $\mathrm{k}(W)=2 \mathrm{k}\left(T_{0}\right)-1<\mathrm{k}\left(T_{0}\right)$, a contradiction to the minimality of $\mathrm{k}\left(T_{0}\right)$. Therefore $\min \left\{\frac{\operatorname{ord}\left(g_{1}\right)}{l_{1}}, \frac{\operatorname{ord}\left(g_{2}\right)}{l_{2}}\right\}>2$ and hence

$$
g_{1}^{\operatorname{ord}\left(g_{1}\right)} \cdot g_{2}^{\operatorname{ord}\left(g_{2}\right)}=T_{0}^{2} \cdot V \text { where } V \text { is non-empty zero-sum sequence. }
$$

It follows that $\left|\mathrm{L}\left(g_{1}^{\operatorname{ord}\left(g_{1}\right)} \cdot g_{2}^{\operatorname{ord}\left(g_{2}\right)}\right)\right| \geq 2$.
3. Suppose that $G_{0}$ is a minimal non-half-factorial set. If $S$ has a minimal zero-sum subsequence $A$ with $\mathrm{k}(A) \neq 1$, then the assertion follows by Lemma 2.2. If $G_{0}$ is an LCN-set, then the assertion follows from 1. and Lemma 2.2.2. Therefore we can suppose $\mathrm{L}(S)=\{\mathrm{k}(S)\}$ and suppose there exists a minimal zero-sum sequence $T$ over $G_{0}$ with $\mathrm{k}(T)<1$.

Let $T_{0}=\prod_{i=1}^{\left|G_{0}\right|} g_{i}^{l_{i}}$ be the minimal zero-sum sequence over $G_{0}$ such that $\mathrm{k}\left(T_{0}\right)$ is minimal. The minimality of $G_{0}$ implies that $l_{i} \geq 1$ for all $i \in\left[1,\left|G_{0}\right|\right]$. After renumbering if necessary, we let

$$
\frac{\operatorname{ord}\left(g_{1}\right)}{l_{1}}=\min \left\{\left.\frac{\operatorname{ord}\left(g_{i}\right)}{l_{i}} \right\rvert\, i \in\left[1,\left|G_{0}\right|\right]\right\}
$$

By Lemma 2.2.3, $\left|\mathrm{L}\left(T_{0}^{\left\lceil\frac{\operatorname{ord}\left(g_{1}\right)}{l_{1}}\right\rceil}\right)\right| \geq 2$. If $T_{0}^{\left\lceil\frac{\operatorname{ord}\left(g_{1}\right)}{l_{1}}\right\rceil} \operatorname{divides} S^{\exp (G)}$, the assertion follows by Lemma 2.2.2. Suppose $T_{0}^{\left\lceil\frac{\operatorname{ord}\left(g_{1}\right)}{l_{1}}\right\rceil} \nmid S^{\exp (G)}$. Let

$$
I=\left\{i \in\left[1,\left|G_{0}\right|\right] \left\lvert\,\left\lceil\frac{\operatorname{ord}\left(g_{1}\right)}{l_{1}}\right\rceil l_{i}>\exp (G) \mathbf{v}_{g_{i}}(S)\right.\right\}
$$

Thus for each $i \in I$, we have

$$
2 \operatorname{ord}\left(g_{i}\right)>l_{i}\left\lceil\frac{\operatorname{ord}\left(g_{i}\right)}{l_{i}}\right\rceil \geq l_{i}\left\lceil\frac{\operatorname{ord}\left(g_{1}\right)}{l_{1}}\right\rceil>\exp (G) \mathrm{v}_{g_{i}}(S) \geq \exp (G)
$$

which implies that $\operatorname{ord}\left(g_{i}\right)=\exp (G), \mathrm{v}_{g_{i}}(S)=1$, and $\left\lceil\frac{\operatorname{ord}\left(g_{1}\right)}{l_{1}}\right\rceil>\frac{\operatorname{ord}\left(g_{i}\right)}{l_{i}}=\frac{\exp (G)}{l_{i}}$.

Let $i_{0} \in I$ such that $l_{i_{0}}=\max \left\{l_{i} \mid i \in I\right\}$. Therefore for every $j \in\left[1,\left|G_{0}\right|\right] \backslash I$, we have

$$
l_{j} \leq \frac{\exp (G) \mathrm{v}_{g_{j}}(S)}{\left\lceil\frac{\operatorname{ord}\left(g_{1}\right)}{l_{1}}\right\rceil} \leq \frac{\exp (G) \mathrm{v}_{g_{j}}(S)}{\frac{\exp (G)}{l_{i_{0}}}}=l_{i_{0}} \vee_{g_{j}}(S)
$$

Note that for every $i \in I$, we have $l_{i} \leq l_{i_{0}}=l_{i_{0}} \mathrm{v}_{g_{i}}(S)$. It follows by $\mathrm{v}_{g_{i_{0}}}\left(T_{0}\right)=l_{i_{0}}=$ $l_{i_{0}} v_{g_{i_{0}}}(S)=\mathrm{v}_{g_{i_{0}}}\left(S^{l_{i_{0}}}\right)$ that

$$
S^{l_{i}}=T_{0} \cdot W, \text { where } W \text { is a zero-sum sequence over } G_{0} \backslash\left\{g_{i_{0}}\right\} .
$$

By the minimality of $G_{0}$, we have $G_{0} \backslash\left\{g_{i_{0}}\right\}$ is half-factorial which implies that $\mathrm{k}(W) \in \mathbb{N}$. Therefore $\mathrm{k}\left(T_{0}\right)=l_{i_{0}} \mathrm{k}(S)-\mathrm{k}(W)$ is an integer, a contradiction to $\mathrm{k}\left(T_{0}\right)<1$.
4. Let $G_{1}=\left\{g \in G_{0} \mid \operatorname{ord}(g)=\exp (G) \mathrm{v}_{g}(S)\right\}$. Suppose $G_{0} \backslash G_{1}$ is not half-factorial. If $G_{0} \backslash G_{1}$ is an LCN-set, then the assertion follows by Proposition 3.1.1 and Lemma 2.2.2. Otherwise there exits a minimal zero-sum sequence $A$ over $G_{0} \backslash G_{1}$ such that $\mathrm{k}(A)<1$. We may assume that $\mathrm{k}(A)$ is minimal over all minimal zero-sum sequences over $G_{0} \backslash G_{1}$ and that $\min \left\{\left.\frac{\operatorname{ord}(g)}{\mathrm{v}_{g}(A)} \right\rvert\, g \in \operatorname{supp}(A)\right\}=\frac{\operatorname{ord}\left(g_{0}\right)}{\mathrm{v}_{0}(A)}$ for some $g_{0} \in \operatorname{supp}(A) \subset G_{0} \backslash G_{1}$. Thus by Lemma 2.2.3, we have $\left|\mathrm{L}\left(A^{\left\lceil\frac{\operatorname{ord}\left(g_{0}\right)}{\lg _{0}(A)}\right\rceil}\right)\right| \geq 2$. The definition of $G_{1}$ implies that

$$
A^{\left\lceil\frac{\operatorname{ord}\left(g_{0}\right)}{\lg _{0}(A)}\right\rceil} \quad \text { divides } \quad S^{\exp (G)}
$$

and hence the assertion follows.
Suppose $G_{0} \backslash G_{1}$ is half-factorial. Then $G_{1}$ is non-empty and hence $G_{1}=\left\{g_{0}\right\}$ for some $g_{0} \in G_{0}$. If $G_{0}$ is an LCN-set, then the assertion follows by Proposition 3.1.1 and Lemma 2.2.2. Otherwise there exits a minimal zero-sum sequence $A$ over $G_{0}$ such that $\mathrm{k}(A)<1$. We may assume that $\mathrm{k}(A)$ is minimal over all minimal zero-sum sequences over $G_{0}$ and that $\min \left\{\left.\frac{\operatorname{ord}(g)}{\mathrm{V}_{g}(A)} \right\rvert\, g \in \operatorname{supp}(A)\right\}=\frac{\operatorname{ord}\left(g_{1}\right)}{\operatorname{vg}_{g_{1}}(A)}$ for some $g_{1} \in \operatorname{supp}(A) \subset G_{0}$. Thus by Lemma 2.2.3, we have $\left|\mathrm{L}\left(A^{\left\lceil\frac{\operatorname{ord}\left(g_{1}\right)}{\mathrm{v}_{1}(A)}\right\rceil}\right)\right| \geq 2$. For every $g \in G_{0} \backslash G_{1}$, we obtain

$$
\mathrm{v}_{g}(A)\left\lceil\frac{\operatorname{ord}\left(g_{1}\right)}{\mathrm{v}_{g_{1}}(A)}\right\rceil \leq \mathrm{v}_{g}(A)\left\lceil\frac{\operatorname{ord}(g)}{\mathrm{v}_{g}(A)}\right\rceil<2 \operatorname{ord}(g) \leq \exp (G) \mathrm{v}_{g}(S)
$$

If $\mathrm{v}_{g_{0}}(A)\left\lceil\frac{\operatorname{ord}\left(g_{1}\right)}{\mathrm{v}_{g_{1}}(A)}\right\rceil \leq \operatorname{ord}\left(g_{0}\right)=\exp (G)$, then

$$
A^{\left\lceil\frac{\operatorname{ord}\left(g_{1}\right)}{\operatorname{vg}_{1}(A)}\right\rceil} \quad \text { divides } \quad S^{\exp (G)}
$$

and hence $\left|\mathrm{L}\left(S^{\exp (G)}\right)\right| \geq 2$.
Otherwise for every $g \in G \backslash G_{1}$, we have

$$
\frac{\exp (G)}{\mathrm{v}_{g_{0}}(A)}<\left\lceil\frac{\operatorname{ord}\left(g_{1}\right)}{\mathrm{v}_{g_{1}}(A)}\right\rceil \leq\left\lceil\frac{\operatorname{ord}(g)}{\mathrm{v}_{g}(A)}\right\rceil \leq\left\lceil\frac{\exp (G) \mathrm{v}_{g}(S)}{2 \mathrm{v}_{g}(A)}\right\rceil \leq \frac{\exp (G) \mathrm{v}_{g}(S)}{\mathrm{v}_{g}(A)}
$$

Therefore $\mathrm{v}_{g}(A)<\mathrm{v}_{g_{0}}(A) \mathrm{v}_{g}(S)$ for all $g \in G_{0} \backslash G_{1}$ which implies that $A$ divides $S^{\mathrm{v}_{g_{0}}(A)}$. Thus there exits a zero-sum sequence $W$ over $G_{0} \backslash G_{1}$ such that $S^{v_{0}(A)}=A \cdot W$. Since
$G_{0} \backslash G_{1}$ is half-factorial, we obtain $\mathrm{k}(A)=\mathrm{v}_{g_{0}}(A) \mathrm{k}(S)-\mathrm{k}(W)$ is an integer, a contradiction to $k(A)<1$.

Proof of Theorem 1.1. By the definition of transfer Krull monoid, it suffices to prove the assertions for $H=\mathcal{B}\left(G_{0}\right)$ and hence $H$ is half-factorial if and only if $G_{0}$ is half-factorial. If $G_{0}$ is half-factorial, it is easy to see that $\operatorname{hf}\left(G_{0}\right)=1$ and $\left|\mathrm{L}\left(\prod_{g \in G_{0}} g^{2 \operatorname{ord}(g)}\right)\right|=1$. Therefore we only need to show that (b) implies (c) and that (d) implies (c).
(b) $\Rightarrow$ (c) Suppose $\mathrm{hf}\left(G_{0}\right)=1$ and assume to the contrary that $G_{0}$ is not half-factorial. Then there exists $A \in \mathcal{A}\left(G_{0}\right)$ such that $\mathrm{k}(A) \neq 1$, whence $\operatorname{supp}(A)$ is not half-factorial. Therefore $\operatorname{hf}(\operatorname{supp}(A)) \geq 2$, a contradiction.
$(\mathrm{d}) \Rightarrow(\mathrm{c})$ Suppose $\left|\mathrm{L}\left(\prod_{g \in G_{0}} g^{2 \operatorname{ord}(g)}\right)\right|=1$ and assume to the contrary that $G_{0}$ is not halffactorial. If $G_{0}$ is an LCN set, then Proposition 3.1.1 implies that $\left|\mathrm{L}\left(\prod_{g \in G_{0}} g^{\operatorname{ord}(g)}\right)\right| \geq 2$, a contradiction. Thus there exists an atom $A \in \mathcal{A}\left(G_{0}\right)$ with $\mathrm{k}(A)<1$ and we may assume that $\mathrm{k}(A)$ is minimal over all atoms of $\mathcal{B}\left(G_{0}\right)$. Let $g_{0} \in \operatorname{supp}(A)$. Then by Lemma 2.2.3, we have $\left|\mathrm{L}\left(A^{\left\lceil\frac{\operatorname{ord}\left(g_{0}\right)}{\operatorname{vg}_{0}(A)}\right\rceil}\right)\right| \geq 2$, a contradiction to $\left.A^{\left\lceil\frac{\operatorname{ord}\left(g_{0}\right)}{\operatorname{vg}_{0}(A)}\right\rceil} \right\rvert\, \prod_{g \in G_{0}} g^{2 \operatorname{ord}(g)}$.

Proof of Theorem 1.2. By the definition of transfer Krull monoid, it sufficient to prove all assertions for $H=\mathcal{B}(G)$.

1. Suppose $\exp (G)<\infty$. If $|G| \geq 3$, then 2 . implies that $\mathrm{hf}(G)<\infty$. If $|G| \leq 2$, then $\mathcal{B}(G)$ is half-factorial and hence $\mathrm{hf}(G)=1$.

Suppose $\exp (G)=\infty$. If there exists an element $g \in G$ with $\operatorname{ord}(g)=\infty$, then $A_{n}=$ $((n+1) g)(-n g)(-g)$ is an atom for every $n \in \mathbb{N}$. Since $\{(n+1) g,-n g,-g\}$ is not halffactorial and $\left|\mathrm{L}\left(A_{n}^{n}\right)\right|=1$ for every $n \geq 2$, we obtain that $\operatorname{hf}(G) \geq n$ for every $n \geq 2$, that is, $\operatorname{hf}(G)=\infty$. Otherwise $G$ is torsion. Then there exists a sequence $\left(g_{i}\right)_{i=1}^{\infty}$ with $g_{i} \in G$ and $\lim _{i \rightarrow \infty} \operatorname{ord}\left(g_{i}\right)=\infty$. It follows by 1. that $\operatorname{hf}(G) \geq \operatorname{hf}\left(\left\langle g_{i}\right\rangle\right) \geq \operatorname{ord}\left(g_{i}\right)$ for all $i \in \mathbb{N}$, that is, $h f(G)=\infty$.
2. If $G$ is an elementary 2-group and $e_{1}, e_{2}$ are two independent elements, then $\left\{e_{1}, e_{2}, e_{1}+\right.$ $\left.e_{2}\right\}$ is not a half-factorial set and $\left|\mathrm{L}\left(e_{1} e_{2}\left(e_{1}+e_{2}\right)\right)\right|=1$ which implies that $\operatorname{hf}(G) \geq 2=$ $\exp (G)$. Otherwise there exists an element $g \in G$ with $\operatorname{ord}(g)=\exp (G) \geq 3$. Since $\{g,-g\}$ is not half-factorial and $\left|\mathrm{L}\left(g^{\operatorname{ord}(g)-1}(-g)^{\operatorname{ord}(g)-1}\right)\right|=1$, we obtain $\mathrm{hf}(G) \geq \operatorname{ord}(g)=\exp (G)$.

Let $S$ be a zero-sum sequence over $G$ such that $\operatorname{supp}(S)$ is not half-factorial. In order to prove $h f(G) \leq\left\lfloor\frac{3 \exp (G)-3}{2}\right\rfloor$, we show that

$$
\left|\mathrm{L}\left(S\left\lfloor\frac{3 \exp (G)-3}{2}\right\rfloor\right)\right| \geq 2
$$

Set $G_{0}=\operatorname{supp}(S)$. If $G_{0}$ is an LCN-set, the assertion follows by Proposition 3.1.1. Suppose there exists an atom $A \in \mathcal{A}\left(G_{0}\right)$ with $\mathrm{k}(A)<1$. Let $A_{0} \in \mathcal{A}(\operatorname{supp}(S))$ be such that $\mathrm{k}\left(A_{0}\right)$ is minimal over all minimal zero-sum sequences over $G_{0}$ and set $A_{0}=g_{1}^{l_{1}} \cdot \ldots \cdot g_{y}^{l_{y}}$, where $y, l_{1}, \ldots l_{y} \in \mathbb{N}$ and $g_{1}, \ldots, g_{y} \in \operatorname{supp}(S)$ are pairwise distinct elements. If there exists $j \in[1, y]$ such that $2 l_{j} \geq \operatorname{ord}\left(g_{j}\right)$, then $g_{j}^{\operatorname{ord}\left(g_{i}\right)}$ divides $A_{0}^{2}$ and hence $A_{0}^{2}=g_{j}^{\operatorname{ord}\left(g_{j}\right)} \cdot W$
for some non-empty sequence $W \in \mathcal{B}(\operatorname{supp}(S))$. Thus $\mathrm{k}(W)=2 \mathrm{k}\left(A_{0}\right)-1<\mathrm{k}\left(A_{0}\right)$, a contradiction to the minimality of $\mathrm{k}\left(A_{0}\right)$. Therefore

$$
2 l_{i} \leq \operatorname{ord}\left(g_{i}\right)-1 \text { for all } i \in[1, y]
$$

After renumbering if necessary, we assume $\frac{\operatorname{ord}\left(g_{1}\right)}{l_{1}}=\min \left\{\left.\frac{\operatorname{ord}\left(g_{i}\right)}{l_{i}} \right\rvert\, i \in[1, y]\right\}$. Then
$l_{i}\left\lceil\frac{\operatorname{ord}\left(g_{1}\right)}{l_{1}}\right\rceil \leq l_{i}\left\lceil\frac{\operatorname{ord}\left(g_{i}\right)}{l_{i}}\right\rceil \leq l_{i} \frac{\operatorname{ord}\left(g_{i}\right)+l_{i}-1}{l_{i}} \leq \operatorname{ord}\left(g_{i}\right)+\frac{\operatorname{ord}\left(g_{i}\right)-1}{2}-1 \leq \frac{3 \exp (G)-3}{2}$,
which implies $A_{0}^{\left\lceil\frac{\operatorname{ord}\left(g_{1}\right)}{l_{1}}\right\rceil}$ divides $\left.S \frac{3 \exp (G)-3}{2}\right\rfloor^{\lfloor\text {. The assertion follows by Lemma 2.2.3. }}$
3(a). Suppose that $G$ is cyclic and that $g \in G$ with $\operatorname{ord}(g)=|G| \geq 3$. We will show that $\mathrm{hf}(G)=\exp (G)$.

Let $S$ be a zero-sum sequence over $G$ such that $\operatorname{supp}(G)$ is not half-factorial. It suffices to show $\left|\mathrm{L}\left(S^{\exp (G)}\right)\right| \geq 2$. If $\left|\left\{g \in \operatorname{supp}(S)\left|\operatorname{ord}(g)=|G| \mathrm{v}_{g}(S)\right\} \mid \leq 1\right.\right.$, then the assertion follows from Proposition 3.1.4. Suppose $\left|\left\{g \in \operatorname{supp}(S)\left|\operatorname{ord}(g)=|G| \mathrm{v}_{g}(S)\right\} \mid \geq 2\right.\right.$. Then there exist distinct $g_{1}, g_{2} \in \operatorname{supp}(S)$ such that $\operatorname{ord}\left(g_{1}\right)=\operatorname{ord}\left(g_{2}\right)=|G|$. We may assume that $g_{1}=k g_{2}$ for some $k \in \mathbb{N}_{\geq 2}$ with $\operatorname{gcd}(k,|G|)=1$. It follows by $\mathrm{k}\left(g_{1}^{|G|-k} \cdot g_{2}\right)<1$ that $G_{0}=\left\{g_{1}, g_{2}\right\}=\left\{g_{1}, k g_{1}\right\}$ is not half-factorial. By Proposition 3.1.2, we obtain that $\left|\mathrm{L}\left(S^{\exp (G)}\right)\right| \geq 2$.

3(b). Suppose $G$ is a finite abelian group with $\exp (G) \leq 6$. We need to prove that $\operatorname{hf}(G)=\exp (G)$. Let $S$ be a zero-sum sequence over $G$ such that $\operatorname{supp}(G)$ is not halffactorial. It suffices to show $\left|\mathrm{L}\left(S^{\exp (G)}\right)\right| \geq 2$.

If $\operatorname{supp}(S)$ is an LCN-set, the assertion follows by Proposition 3.1.1. Thus there is a minimal zero-sum sequence $W$ over $\operatorname{supp}(S)$ such that

$$
\mathrm{k}(W)<1
$$

By Proposition 3.1.2 and Lemma 2.2.2, we have

$$
|\operatorname{supp}(W)| \geq 3
$$

Suppose $W \mid S$. Since $\mathrm{k}(W)<1$, it follows by Lemma 2.2 that $\left|\mathrm{L}\left(W^{\exp (G)}\right)\right| \geq 2$ and hence $\left|\mathrm{L}\left(S^{\exp (G)}\right)\right| \geq 2$. Therefore we may assume that $W \nmid S$, whence $|W| \geq|\operatorname{supp}(W)|+1 \geq 4$. It follows that $6 \geq \exp (G) \geq \frac{|W|}{\mathrm{k}(W)}>|W| \geq 4$.

We distinguish two cases according to $\exp (G) \in\{5,6\}$.
Case 1. $\exp (G)=5$.
Then, $G \cong C_{5}^{r}$ and for all $W \in \mathcal{A}(\operatorname{supp}(S))$ with $\mathrm{k}(W)<1$, we have that

$$
W \nmid S,|\operatorname{supp}(W)|=3, \quad \text { and }|W|=4 .
$$

Let $W_{0}$ be an atom over $\operatorname{supp}(S)$ with $\mathrm{k}\left(W_{0}\right)<1$. Then $W_{0}$ and $S$ must be of the forms

$$
W_{0}=g_{1}^{2} g_{2} g_{3}, \quad S=T g_{1} g_{2} g_{3}
$$

where $g_{1}, g_{2}, g_{3} \in \operatorname{supp}(S)$ are pairwise distinct and $T \in \mathcal{F}\left(\operatorname{supp}(S) \backslash\left\{g_{1}\right\}\right)$ with $\sigma(T)=g_{1}$.
We may assume $T$ is zero-sum free. Otherwise $T=T_{0} T^{\prime}$ with $T_{0}$ is zero-sum and $T^{\prime}$ is zero-sum free and we can replace $S$ by $T^{\prime} g_{1} g_{2} g_{3}$, since $\left|\mathrm{L}\left(\left(T^{\prime} g_{1} g_{2} g_{3}\right)^{5}\right)\right| \geq 2$ implies that
$\left|\mathrm{L}\left(S^{5}\right)\right| \geq 2$. Therefore $S$ is a product of at most three atoms and every term of $S$ has order 5.

Assume to the contrary that $\left|\mathrm{L}\left(S^{5}\right)\right|=1$, i.e., $\mathrm{L}\left(S^{5}\right)=\{|T|+3\}$. Since $g_{1}^{5} g_{2}^{5} g_{3}^{5}=$ $W_{0}^{2}\left(g_{1} g_{2}^{3} g_{3}^{3}\right)$ is a zero-sum subsequence of $S^{5}$, we obtain that $g_{1} g_{2}^{3} g_{3}^{3}$ is an atom. Note that

$$
\left(g_{1}^{2} g_{2} g_{3}\right)^{2} S=\left(g_{1}^{4} T\right)\left(g_{1} g_{2}^{3} g_{3}^{3}\right) \text { is a zero-sum subsequence of } S^{5}
$$

Suppose $S$ is an atom. Then $\mathrm{L}\left(g_{1}^{4} T\right)=\{2\}$ and hence $\left|g_{1}^{4} T\right| \geq 2 \times 4=8$, i.e., $|T| \geq 4$. It follows by $\{5\}=\mathrm{L}\left(S^{5}\right)=\{|T|+3\}$ that $|T|=2$, a contradiction.

Suppose $S$ is a product of two atoms. Then $\mathrm{L}\left(g_{1}^{4} T\right)=\{3\}$ and hence $\left|g_{1}^{4} T\right| \geq 3 \times 4=12$, i.e., $|T| \geq 8$. It follows by $\{10\}=\mathrm{L}\left(S^{5}\right)=\{|T|+3\}$ that $|T|=7$, a contradiction.

Suppose $S$ is a product of three atoms. Then $\mathrm{L}\left(g_{1}^{4} T\right)=\{4\}$ which implies that $T=$ $T_{1} T_{2} T_{3} T_{4}$ such that $g_{1} T_{i}$ is zero-sum for all $i \in[1,4]$. Since $g_{1} T_{i} \mid S$, we obtain $\mathrm{k}\left(g_{1} T_{i}\right) \geq 1$ and hence $\left|g_{1} T_{i}\right| \geq 5$. Therefore $\left|g_{1}^{4} T\right| \geq 4 \times 5=20$, i.e., $|T| \geq 16$. It follows by $\{15\}=\mathrm{L}\left(S^{5}\right)=\{|T|+3\}$ that $|T|=12$, a contradiction.

Case 2. $\exp (G)=6$.
Let $W$ be an atom over $\operatorname{supp}(S)$ with $\mathrm{k}(W)<1$. If $|W|=4$, then $|\operatorname{supp}(W)|=3$ and hence $W$ must be of the form

$$
W=g_{1}^{2} g_{2} g_{3}
$$

where $g_{1}, g_{2}, g_{3} \in \operatorname{supp}(S)$ are pairwise distinct. Since $\left(g_{1}^{6}\right)\left(g_{2}^{6}\right)\left(g_{3}^{6}\right)=W^{3}\left(g_{2}^{3} g_{3}^{3}\right)$, we obtain that $\left|\mathrm{L}\left(g_{1}^{6} g_{2}^{6} g_{3}^{6}\right)\right| \geq 2$. It follows by $g_{1}^{6} g_{2}^{6} g_{3}^{6}$ divides $S^{\exp (G)}$ that $\left|\mathrm{L}\left(S^{\exp (G)}\right)\right| \geq 2$.

If $|W|=5$, then $|\operatorname{supp}(W)|=3$ or 4 and hence $W$ must be one of the following forms.
i. $W=g_{1}^{3} g_{2} g_{3}$ with $g_{1}, g_{2}, g_{3} \in \operatorname{supp}(S)$ are pairwise distinct.
ii. $W=g_{1}^{2} g_{2}^{2} g_{3}$ with $g_{1}, g_{2}, g_{3} \in \operatorname{supp}(S)$ are pairwise distinct.
iii. $W=g_{1}^{2} g_{2} g_{3} g_{4}$ with $g_{1}, g_{2}, g_{3}, g_{4} \in \operatorname{supp}(S)$ are pairwise distinct.

Suppose (i) holds. Then $0=2 \sigma(W)=6 g_{1}+2 g_{2}+2 g_{3}=2 g_{2}+2 g_{3}$. Since $\left(g_{1}^{6}\right)\left(g_{2}^{6}\right)\left(g_{3}^{6}\right)=$ $W^{2}\left(g_{2}^{2} g_{3}^{2}\right)^{2}$, we obtain that $\left|\mathrm{L}\left(g_{1}^{6} g_{2}^{6} g_{3}^{6}\right)\right| \geq 2$. It follows by $g_{1}^{6} g_{2}^{6} g_{3}^{6}$ divides $S^{\exp (G)}$ that $\left|\mathrm{L}\left(S^{\exp (G)}\right)\right| \geq 2$.

Suppose (ii) holds. Then $0=3 \sigma(W)=6 g_{1}+6 g_{2}+3 g_{3}=3 g_{3}$. Thus ord $\left(g_{3}\right)=2$ and hence $\mathrm{k}(W) \geq 1 / 2+4 / 6>1$, a contradiction.

Suppose (iii) holds. Then $0=3 \sigma(W)=6 g_{1}+3 g_{3}+3 g_{3}+3 g_{4}=3 g_{2}+3 g_{3}+3 g_{4}$. Therefore $W_{0}=g_{2}^{3} g_{3}^{3} g_{4}^{3}$ is zero-sum. If $W_{0}=g_{1}^{6} g_{2}^{6} g_{3}^{6} g_{4}^{6}(W)^{-3}$ is not a minimal zero-sum sequence, then $\left|\mathrm{L}\left(g_{1}^{6} g_{2}^{6} g_{3}^{6} g_{4}^{6}\right)\right| \geq 2$ and hence $\left|\mathrm{L}\left(S^{\exp (G)}\right)\right| \geq 2$. If $W_{0}$ is minimal zero-sum, then $g_{1}^{6} g_{2}^{6} g_{3}^{6} g_{4}^{6}=\left(g_{1}^{6}\right) W_{0}^{2}$ implies that $\left|\mathrm{L}\left(g_{1}^{6} g_{2}^{6} g_{3}^{6} g_{4}^{6}\right)\right| \geq 2$ and hence $\left|\mathrm{L}\left(S^{\exp (G)}\right)\right| \geq 2$.

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