ON HALF-FACTORIALITY OF TRANSFER KRULL MONOIDS

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ABSTRACT. Let H be a transfer Krull monoid over a subset G_0 of an abelian group G with finite exponent. Then every non-unit $a \in H$ can be written as a finite product of atoms, say $a = u_1 \cdot \ldots \cdot u_k$. The set $\mathsf{L}(a)$ of all possible factorization lengths k is called the set of lengths of a, and H is said to be half-factorial if $|\mathsf{L}(a)| = 1$ for all $a \in H$.

We show that, if $a \in H$ and $|\mathsf{L}(a^{\lfloor(3\exp(G)-3)/2\rfloor})| = 1$, then the smallest divisor-closed submonoid of H containing a is half-factorial. In addition, we prove that, if G_0 is finite and $|\mathsf{L}(\prod_{g\in G_0} g^{2\operatorname{ord}(g)})| = 1$, then H is half-factorial.

1. INTRODUCTION

Let H be a monoid. If an element $a \in H$ has a factorization $a = u_1 \cdot \ldots \cdot u_k$, where $k \in \mathbb{N}$ and u_1, \ldots, u_k are atoms of H, then k is called a factorization length of a, and the set $\mathsf{L}(a)$ of all possible k is referred to as the set of lengths of a. The monoid H is said to be half-factorial (half-factorial) if $|\mathsf{L}(a)| = 1$ for every $a \in H$. Half-factoriality has been a central topic in factorization theory since the early days of this field (e.g., see [3, 4, 15, 19, 5, 11, 16]).

Given $a \in H$, let $\llbracket a \rrbracket = \{b \in H \mid b \text{ divides some power of } a\}$ be the smallest divisorclosed submonoid of H containing a. Then $\llbracket a \rrbracket$ is half-factorial if and only if $|\mathsf{L}(a^n)| = 1$ for all $n \in \mathbb{N}$, and H is half-factorial if and only if $\llbracket c \rrbracket$ is half-factorial for every $c \in H$. It is thus natural to ask:

Does there exist an integer $N \in \mathbb{N}$ such that, if $a \in H$ and $|\mathsf{L}(a^N)| = 1$, then $[\![a]\!]$ is half-factorial? (Note that, if $[\![a]\!]$ is half-factorial for some $a \in H$, then of course $|\mathsf{L}(a^k)| = 1$ for every $k \in \mathbb{N}$.)

We answer this question affirmatively for transfer Krull monoids over finite abelian groups, and we study the smallest N having the above property (Theorems 1.1 and 1.2).

Transfer Krull monoids and transfer Krull domains are a recently introduced class of monoids and domains including, among others, all commutative Krull domains and wide classes of non-commutative Dedekind domains (see Section 2 and [8] for a survey).

Let H be a transfer Krull monoid over a subset G_0 of an abelian group G. Then H is half-factorial if and only if the monoid $\mathcal{B}(G_0)$ of zero-sum sequences over G_0 is half-factorial (in this case, we also say that the set G_0 is half-factorial). It is a standing conjecture that

²⁰¹⁰ Mathematics Subject Classification. 11B30, 11R27, 13A05, 13F05, 20M13.

Key words and phrases. Transfer Krull monoids, zero-sum sequences, sets of lengths, half-factorial.

The last-named author was supported by the Austrian Science Fund (FWF Project P28864–N35).

every abelian group has a half-factorial generating set, which implies that every abelian group can be realized as the class group of a half-factorial Dedekind domain ([9]).

Suppose now that H is a commutative Krull monoid with class group G and that every class contains a prime divisor. It is a classic result that H is half-factorial if and only if $|G| \leq 2$, and it turns out that, also for $|G| \geq 3$, half-factorial subsets (and minimal nonhalf-factorial subsets) of the class group G play a crucial role in a variety of arithmetical questions (see [10, Chapter 6.7], [14]). However, we are far away from a good understanding of half-factorial sets in finite abelian groups (see [21] for a survey, and [17, 18, 22]). To mention one open question, the maximal size of half-factorial subsets is unknown even for finite cyclic groups ([18]). Our results open the door to a computational approach to the study of half-factorial sets.

More in detail, denote by hf(H) the infimum of all $N \in \mathbb{N}$ with the following property:

If
$$a \in H$$
 and $|\mathsf{L}(a^N)| = 1$, then $[a]$ is half-factorial.

(Here, as usual, we assume $\inf \emptyset = \infty$.) We call hf(H) the half-factoriality index of H. If H is not half-factorial, then hf(H) is the infimum of all $N \in \mathbb{N}$ with the property that $|\mathsf{L}(a^N)| \geq 2$ for every $a \in H$ such that $[\![a]\!]$ is not half-factorial.

Theorem 1.1. Let H be a transfer Krull monoid over a finite subset G_0 of an abelian group G with finite exponent. The following are equivalent:

(a) *H* is half-factorial.

(b) hf(H) = 1.

- (c) G_0 is half-factorial.
- (d) $\left| \mathsf{L} \left(\prod_{q \in G_0} g^{2 \operatorname{ord}(g)} \right) \right| = 1.$

We observe that in general if H is half-factorial, then hf(H) = 1. But if H is a transfer Krull monoid over a subset of a torsion free group, then hf(H) = 1 does not imply that H is half-factorial (see Example 2.4.1). Furthermore, for every $n \in \mathbb{N}$, there exists a Krull monoid H with finite class group such that hf(H) = n (see Example 2.4.2).

Theorem 1.2. Let H be a transfer Krull monoid over an abelian group G.

- 1. $hf(H) < \infty$ if and only if $exp(G) < \infty$.
- 2. If $\exp(G) < \infty$ and $|G| \ge 3$, then $\exp(G) \le hf(H) \le \frac{3}{2}(\exp(G) 1)$.
- 3. If G is cyclic or $\exp(G) \leq 6$, then $hf(H) = \exp(G)$.

We postpone the proofs of Theorems 1.1 and 1.2 to Section 3.

2. Preliminaries

Our notation and terminology are consistent with [10]. Let \mathbb{N} be the set of positive integers, let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and let \mathbb{Q} be the set of rational numbers. For integers $a, b \in \mathbb{Z}$, we denote by $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$ the discrete, finite interval between a and b.

Atomic monoids. By a monoid, we mean an associative semigroup with identity, and if not stated otherwise we use multiplicative notation. Let H be a monoid with identity $1 = 1_H \in H$. An element $a \in H$ is said to be invertible (or a unit) if there exists an element

 $a' \in H$ such that aa' = a'a = 1. The set of invertible elements of H will be denoted by H^{\times} , and we say that H is reduced if $H^{\times} = \{1\}$. The monoid H is said to be unit-cancellative if for any two elements $a, u \in H$, each of the equations au = a or ua = a implies that $u \in H^{\times}$. Clearly, every cancellative monoid is unit-cancellative.

Suppose that H is unit-cancellative. An element $u \in H$ is said to be irreducible (or an atom) if $u \notin H^{\times}$ and for any two elements $a, b \in H, u = ab$ implies that $a \in H^{\times}$ or $b \in H^{\times}$. Let $\mathcal{A}(H)$ denote the set of atoms of H. We say that H is atomic if every non-unit is a finite product of atoms. If H satisfies the ascending chain condition on principal left ideals and on principal right ideals, then H is atomic ([7, Theorem 2.6]). If $a \in H \setminus H^{\times}$ and $a = u_1 \dots u_k$, where $k \in \mathbb{N}$ and $u_1, \dots, u_k \in \mathcal{A}(H)$, then k is a factorization length of a, and

$$\mathsf{L}_{H}(a) = \mathsf{L}(a) = \{k \in \mathbb{N} \mid k \text{ is a factorization length of } a\}$$

denotes the set of lengths of a. It is convenient to set $L(a) = \{0\}$ for all $a \in H^{\times}$.

A transfer Krull mononid is a monoid H having a weak transfer homomorphism (in the sense of [2, Definition 2.1]) $\theta: H \to \mathcal{B}(G_0)$, where $\mathcal{B}(G_0)$ is the monoid of zero-sum sequences over a subset G_0 of an abelian group G. If H is a commutative Krull monoid with class group G and $G_0 \subset G$ is the set of classes containing prime divisors, then there is a weak transfer homomorphism $\theta: H \to \mathcal{B}(G_0)$. Beyond that, there are wide classes of non-commutative Dedekind domains having a weak transfer homomorphism to a monoid of zero-sum sequences ([25, Theorem 1.1], [24, Theorem 4.4]). We refer to [8, 13] for surveys on transfer Krull monoids. If $\theta: H \to \mathcal{B}(G_0)$ is a weak transfer homomorphism, then sets of lengths in H and in $\mathcal{B}(G_0)$ coincide ([2, Lemma 2.7]) and thus the statements of Theorems 1.1 and 1.2 can be proved in the setting of monoids of zero-sum sequences.

Monoids of zero-sum sequences. Let G be an abelian group and let $G_0 \subset G$ be a non-empty subset. Then $\langle G_0 \rangle$ denotes the subgroup generated by G_0 . In Additive Combinatorics, a sequence (over G_0) means a finite unordered sequence of terms from G_0 where repetition is allowed, and (as usual) we consider sequences as elements of the free abelian monoid with basis G_0 . Let

$$S = g_1 \cdot \ldots \cdot g_\ell = \prod_{g \in G_0} g^{\mathsf{v}_g(S)} \in \mathcal{F}(G_0)$$

be a sequence over G_0 . We call

$$\begin{split} \operatorname{supp}(S) &= \{g \in G \mid \mathsf{v}_g(S) > 0\} \subset G \text{ the } support \text{ of } S, \\ |S| &= \ell = \sum_{g \in G} \mathsf{v}_g(S) \in \mathbb{N}_0 \text{ the } length \text{ of } S, \\ \sigma(S) &= \sum_{i=1}^{\ell} g_i \text{ the } sum \text{ of } S, \\ \operatorname{and} \quad \Sigma(S) &= \Big\{ \sum_{i \in I} g_i \mid \emptyset \neq I \subset [1, \ell] \Big\} \text{ the } set of subsequence sums} \end{split}$$

The sequence S is said to be

of S.

- zero-sum free if $0 \notin \Sigma(S)$,
- a zero-sum sequence if $\sigma(S) = 0$,
- a *minimal zero-sum sequence* if it is a nontrivial zero-sum sequence and every proper subsequence is zero-sum free.

The set of zero-sum sequences $\mathcal{B}(G_0) = \{S \in \mathcal{F}(G_0) \mid \sigma(S) = 0\} \subset \mathcal{F}(G_0)$ is a submonoid, and the set of minimal zero-sum sequences is the set of atoms of $\mathcal{B}(G_0)$. For any arithmetical invariant *(H) defined for a monoid H, we write $*(G_0)$ instead of $*(\mathcal{B}(G_0))$. In particular, $\mathcal{A}(G_0) = \mathcal{A}(\mathcal{B}(G_0))$ is the set of atoms of $\mathcal{B}(G_0)$ and $hf(G_0) = hf(\mathcal{B}(G_0))$.

Let G be an abelian group. We denote by $\exp(G)$ the exponent of G which is the least common multiple of the orders of all elements of G. Let $r \in \mathbb{N}$ and let (e_1, \ldots, e_r) be an r-tuple of elements of G. Then (e_1, \ldots, e_r) is said to be independent if $e_i \neq 0$ for all $i \in [1, r]$ and if for all $(m_1, \ldots, m_r) \in \mathbb{Z}^r$ an equation $m_1e_1 + \ldots + m_re_r = 0$ implies that $m_ie_i = 0$ for all $i \in [1, r]$. Suppose G is finite. The r-tuple (e_1, \ldots, e_r) is said to be a basis of G if it is independent and $G = \langle e_1 \rangle \oplus \ldots \oplus \langle e_r \rangle$. For every $n \in \mathbb{N}$, we denote by C_n an additive cyclic group of order n. Since $G \cong C_{n_1} \oplus \ldots \oplus C_{n_r}$, $r = \mathsf{r}(G)$ is the rank of G and $n_r = \exp(G)$ is the exponent of G.

Let $G_0 \subset G$ be a non-empty subset. For a sequence $S = g_1 \cdot \ldots \cdot g_\ell \in \mathcal{F}(G_0)$, we call

$$\mathsf{k}(S) = \sum_{i=1}^{l} \frac{1}{\operatorname{ord}(g_i)} \in \mathbb{Q}_{\geq 0} \quad \text{the cross number of } S, \text{ and}$$
$$\mathsf{K}(G_0) = \max\{\mathsf{k}(S) \mid S \in \mathcal{A}(G_0)\} \quad \text{the cross number of } G_0$$

For the relevance of cross numbers in the theory of non-unique factorizations, see [18, 20, 23] and [10, Chapter 6].

The set G_0 is called

- half-factorial if the monoid $\mathcal{B}(G_0)$ is half-factorial;
- non-half-factorial if the monoid $\mathcal{B}(G_0)$ is not half-factorial;
- minimal non-half-factorial if G_0 is not half-factorial but all its proper subsets are;
- an LCN-set if $k(A) \ge 1$ for all atoms $A \in \mathcal{A}(G_0)$.

The following simple result ([10, Proposition 6.7.3]) will be used throughout the paper without further mention.

Lemma 2.1. Let G be a finite abelian group and $G_0 \subset G$ a subset. Then the following statements are equivalent:

- (a) G_0 is half-factorial.
- (b) $\mathsf{k}(U) = 1$ for every $U \in \mathcal{A}(G_0)$.
- (c) $\mathsf{L}(B) = \{\mathsf{k}(B)\}$ for every $B \in \mathcal{B}(G_0)$.

Lemma 2.2. Let G be a finite group, let $G_0 \subset G$ be a subset, let S be a zero-sum sequence over G_0 , and let A be a minimal zero-sum sequence over G_0 .

- 1. If $k(A) \neq 1$, then $|L(A^{\exp(G)})| \geq 2$.
- 2. If there exists a zero-sum subsequence T of S such that $|\mathsf{L}(T)| \ge 2$, then $|\mathsf{L}(S)| \ge 2$.

$$\left| \mathsf{L}\left(A^{\left\lceil \frac{\operatorname{ord}(g)}{\mathsf{v}_g(A)} \right\rceil} \right) \right| \ge 2, \quad \text{for all } g \in \operatorname{supp}(A).$$

Proof. 1. Suppose $\mathsf{k}(A) \neq 1$ and let $A = g_1 \cdot \ldots \cdot g_\ell$, where $\ell \in \mathbb{N}$ and $g_1, \ldots, g_\ell \in G_0$. Then $A^{\exp(G)} = (g_1^{\operatorname{ord}(g_1)})^{\frac{\exp(G)}{\operatorname{ord}(g_1)}} \cdot \ldots \cdot (g_\ell^{\operatorname{ord}(g_\ell)})^{\frac{\exp(G)}{\operatorname{ord}(g_\ell)}},$

which implies that

$$\left\{\exp(G), \sum_{i=1}^{\ell} \frac{\exp(G)}{\operatorname{ord}(g_i)}\right\} = \left\{\exp(G), \exp(G)\mathsf{k}(A)\right\} \subset \mathsf{L}(A^{\exp(G)}).$$

It follows by $\mathbf{k}(A) \neq 1$ that $|\mathbf{L}(A^{\exp(G)})| \geq 2$.

2. Suppose T is a zero-sum subsequence of S with $|\mathsf{L}(T)| \ge 2$. It follows by $\mathsf{L}(S) \supset \mathsf{L}(T) + \mathsf{L}(ST^{-1})$ that $|\mathsf{L}(S)| \ge |\mathsf{L}(T)| \ge 2$.

3. Suppose $\mathsf{k}(A) < 1$ and $\mathsf{k}(A)$ is minimal over all minimal zero-sum sequences over G_0 . Let $g \in \operatorname{supp}(A)$. Then there exist $s \in \mathbb{N}$ and minimal zero-sum sequences W_1, \ldots, W_s such that

$$A^{\lceil \frac{\operatorname{ord}(g)}{\operatorname{v}_g(A)} \rceil} = g^{\operatorname{ord}(g)} \cdot W_1 \cdot \ldots \cdot W_s$$

Since

we have [

$$\mathsf{k}\left(A^{\left\lceil \frac{\operatorname{ord}(g)}{\operatorname{v}_g(A)} \right\rceil}\right) = \left\lceil \frac{\operatorname{ord}(g)}{\operatorname{v}_g(A)} \right\rceil \mathsf{k}(A) = 1 + \sum_{i=1}^s \mathsf{k}(W_i) > (1+s)\mathsf{k}(A) ,$$

$$\frac{\operatorname{ord}(g)}{\operatorname{v}_g(A)} \rceil \neq s+1 \text{ and hence } \left|\mathsf{L}\left(A^{\left\lceil \frac{\operatorname{ord}(g)}{\operatorname{v}_g(A)} \right\rceil}\right)\right| \ge 2.$$

For commutative and finitely generated monoids, we have the following result.

Proposition 2.3. Let H be a commutative unit-cancellative monoid. If H_{red} is finitely generated, then hf(H) is finite.

Proof. We may assume that H is reduced and not half-factorial. Suppose H is finitely generated and suppose $\mathcal{A}(H) = \{u_1, \ldots, u_n\}$, where $n \in \mathbb{N}$. Set $A_0 = \{\prod_{i \in I} u_i \mid \emptyset \neq I \subset [1, n]\}$. Then A_0 is finite and hence there exists $M \in \mathbb{N}$ such that $|\mathsf{L}(a_0^M)| \geq 2$ for all $a_0 \in A_0$ with $[\![a_0]\!]$ not half-factorial. Let $a \in H \setminus H^{\times}$ such that $[\![a]\!]$ is not half-factorial. It suffices to show that $|\mathsf{L}(a^M)| \geq 2$. Suppose $a = u_1^{k_1} \cdot \ldots \cdot u_n^{k_n}$, where $k_1, \ldots, k_n \in \mathbb{N}_0$. Set $I_0 = \{i \in [1, n] \mid k_i \geq 1\}$ and $a_0 = \prod_{i \in I} u_i$. Then a_0 divides a and $[\![a_0]\!] = [\![a]\!]$ is not half-factorial, whence $|\mathsf{L}(a_0^M)| \geq 2$ and $|\mathsf{L}(a^M)| \geq 2$.

If G_0 is a finite subset of an abelian group, then $\mathcal{B}(G_0)$ is finitely generated ([10, Theorem 3.4.2]) and thus $hf(G_0) < \infty$. We refer to [6, Sections 3.2 and 3.3] and [12] for semigroups of ideals and semigroups of modules that are finitely generated unit-cancellative but not necessarily cancellative.

Examples 2.4. The following examples will help up to illustrate some important points.

- 1. Let (e_1, e_2) be a basis of \mathbb{Z}^2 and let $G_0 = \{e_1, -e_1, e_2, -e_2, e_1 + e_2, -e_1 e_2\}$. Then $\mathcal{A}(G_0) = \{e_1(-e_1), e_2(-e_2), (e_1 + e_2)(-e_1 - e_2), e_1e_2(-e_1 - e_2), (-e_1)(-e_2)(e_1 + e_2)\}$. Since $e_1(-e_1) \cdot e_2(-e_2) \cdot (e_1 + e_2)(-e_1 - e_2) = e_1e_2(-e_1 - e_2) \cdot (-e_1)(-e_2)(e_1 + e_2)$, we obtain G_0 is not half-factorial. Furthermore, we have G_1 is half-factorial for every nonempty proper subset $G_1 \subsetneq G_0$. Let $A \in \mathcal{B}(G_0)$. If $\operatorname{supp}(A) = G_0$, then $|\mathsf{L}(A)| \ge 2$ and $[\![A]\!] = \mathcal{B}(G_0)$ is not half-factorial. If $\operatorname{supp}(A) \subsetneq G_0$, then $[\![A]\!] = \mathcal{B}(\operatorname{supp}(A))$ is half-factorial and $|\mathsf{L}(A)| = 1$. Therefore $hf(G_0) = 1$.
- 2. Let G be a cyclic group with order n and let $g \in G$ with $\operatorname{ord}(g) = n$, where $n \in \mathbb{N}_{\geq 3}$. Set $G_0 = \{g, -g\}$. Then G_0 is not half-factorial. Let $A_0 = g(-g)$. Then $\llbracket A_0 \rrbracket$ is not half-factorial and $|\mathsf{L}(A_0^{n-1})| = 1$, whence $\mathsf{hf}(G_0) \geq n$. Let $A \in \mathcal{B}(G_0)$ with $\llbracket A \rrbracket$ is not half-factorial. Then $\operatorname{supp}(A) = G_0$ and A_0 divides A, whence $|\mathsf{L}(A^n)| \geq 2$. Therefore $\mathsf{hf}(G_0) = n$. Let $G \cong C_2^2$ and let (e_1, e_2) be a basis of G. Set $G_1 = \{e_1, e_2, e_1 + e_2\}$. Then G_1 is not half-factorial. Let $A_1 = e_1e_2(e_1 + e_2)$. Then $\llbracket A_1 \rrbracket$ is not half-factorial and $|\mathsf{L}(A_1)| = 1$, whence $\mathsf{hf}(G_1) \geq 2$. Let $A \in \mathcal{B}(G_1)$ with $\llbracket A \rrbracket$ is not half-factorial. Then $\mathsf{supp}(A) = G_1$ and A_1 divides A, whence $|\mathsf{L}(A^2)| \geq 2$. Therefore $\mathsf{hf}(G_1) = 2$.
- 3. Let *H* be a bifurcus moniod (i.e. $2 \in L(a)$ for all $a \in H \setminus (H^{\times} \cup \mathcal{A}(H))$). For examples, see [1, Examples 2.1 and 2.2]. Since for every $a \in H \setminus H^{\times}$, we have $\{2,3\} \subset L(a^3)$, it follows that $hf(H) \leq 3$ and hf(H) is the minimal integer $t \in \mathbb{N}$ such that $|L(a^t)| \geq 2$ for all $a \in H \setminus H^{\times}$. Therefore hf(H) = 3 if and only if there exists $a_0 \in \mathcal{A}(H)$ such that $L(a_0^2) = \{2\}$.
- 4. Let $H \,\subset F = F^{\times} \times [p_1, \ldots, p_s]$ be a non-half factorial finitely primary monoid of rank s and exponent α (see [10, Definition 2.9.1]). For every $a = \epsilon p_1^{t_1} \ldots p_s^{t_s} \in F$, we define $||a|| = t_1 + \ldots + t_s$, where $t_1, \ldots, t_s \in \mathbb{N}_0$ and $\epsilon \in F^{\times}$. Let $a \in H \setminus H^{\times}$. Since H is primary, we have $H = [\![a]\!]$ is not half-factorial. Thus hf(H) is the minimal integer $t \in \mathbb{N}$ such that $|\mathsf{L}(a^t)| \geq 2$ for all $a \in H \setminus H^{\times}$. Suppose $a_0 \in H$ with $||a_0|| = \min\{||a||: a \in H \setminus H^{\times}\}$. Then $a_0 \in \mathcal{A}(H)$ and $\mathsf{L}(a_0^2) = \{2\}$, whence $hf(H) \geq 3$.

If $H \setminus H^{\times} = (p_1 \dots p_s)^{\alpha} F$ and $s \geq 2$, then H is bifurcus and hence hf(H) = 3. Suppose s = 1 and $H \setminus H^{\times} = (p_1)^{\alpha} F$. Let $b = \epsilon p^{\beta} \in H$. Then $p^{3\alpha}$ divides b^4 . It follows by $p^{3\alpha} = (p^{\alpha})^3 = p^{\alpha+1}p^{2\alpha-1}$ that $|\mathsf{L}(b^4)| \geq 2$, whence $hf(H) \leq 4$. If $3\beta \geq 4\alpha$, then $p^{3\alpha}$ divides b^3 and hence $|\mathsf{L}(b^3)| \geq 2$. If $3\beta \leq 4\alpha - 2$, then b is an atom and $b^3 = \epsilon^3 p^{2\alpha-1} p^{3\beta-(2\alpha-1)}$, whence $|\mathsf{L}(b^3)| \geq 2$. If $3\beta = 4\alpha - 1$, then $\mathsf{L}(b^3) = \{3\}$. Put all together, if $\alpha \equiv 1 \mod 3$, then hf(H) = 4. Otherwise hf(H) = 3.

3. Proof of main theorem

Proposition 3.1. Let $G_0 \subset G$ be a non half-factorial subset and let S be a zero-sum sequence over G_0 with $supp(S) = G_0$.

- 1. If G_0 is an LCN-set, then $|\mathsf{L}(\prod_{g \in G_0} g^{\operatorname{ord}(g)})| \geq 2$.
- 2. If $|G_0| = 2$, then $|\mathsf{L}(\prod_{g \in G_0} g^{\operatorname{ord}(g)})| \ge 2$.
- 3. If G_0 is a minimal non half-factorial subset, then $|\mathsf{L}(S^{\exp(G)})| \geq 2$.
- 4. If $|\{g \in G_0 \mid \operatorname{ord}(g)/\mathsf{v}_q(S) = \exp(G)\}| \le 1$, then $|\mathsf{L}(S^{\exp(G)})| \ge 2$.

Proof. 1. Suppose G_0 is an LCN-set. Since G_0 is not half-factorial, there exists a minimal zero-sum sequence T over G_0 such that $\mathsf{k}(T) > 1$. Note that T is a subsequence of $\prod_{a \in G_0} g^{\operatorname{ord}(g)}$. Then there exists $W_1, \ldots, W_l \in \mathcal{A}(G_0)$ such that

$$\prod_{g \in G_0} g^{\operatorname{ord}(g)} = T \cdot W_1 \cdot \ldots \cdot W_l \,.$$

Thus $\mathsf{k}(\prod_{g \in G_0} g^{\operatorname{ord}(g)}) = |G_0| = \mathsf{k}(T) + \sum_{i=1}^l \mathsf{k}(W_i) > 1 + l$. The assertion follows by $\{|G_0|, 1+l\} \subset \mathsf{L}(\prod_{g \in G_0} g^{\operatorname{ord}(g)}).$

2. Suppose $|G_0| = 2$ and let $G_0 = \{g_1, g_2\}$. If G_0 is an LCN-set, the assertion follows by 1.. Suppose there exists a minimal zero-sum sequence T over G_0 with $\mathsf{k}(T) < 1$. Let $T_0 = g_1^{l_1} \cdot g_2^{l_2}$ be the minimal zero-sum sequence over G_0 such that $\mathsf{k}(T_0)$ is minimal. If $\min\{\frac{\operatorname{ord}(g_1)}{l_1}, \frac{\operatorname{ord}(g_2)}{l_2}\} \leq 2$, say $\frac{\operatorname{ord}(g_1)}{l_1} \leq 2$ then

 $T_0^2 = g_1^{\operatorname{ord}(g_1)} \cdot W$, where W is non-empty zero-sum sequence.

Thus $k(W) = 2k(T_0) - 1 < k(T_0)$, a contradiction to the minimality of $k(T_0)$. Therefore $\min\{\frac{\operatorname{ord}(g_1)}{l_1}, \frac{\operatorname{ord}(g_2)}{l_2}\} > 2$ and hence

 $g_1^{\operatorname{ord}(g_1)} \cdot g_2^{\operatorname{ord}(g_2)} = T_0^2 \cdot V$ where V is non-empty zero-sum sequence.

It follows that $|\mathsf{L}(g_1^{\operatorname{ord}(g_1)} \cdot g_2^{\operatorname{ord}(g_2)})| \ge 2.$

3. Suppose that G_0 is a minimal non-half-factorial set. If S has a minimal zero-sum subsequence A with $k(A) \neq 1$, then the assertion follows by Lemma 2.2. If G_0 is an LCN-set, then the assertion follows from 1. and Lemma 2.2.2. Therefore we can suppose $L(S) = \{k(S)\}$ and suppose there exists a minimal zero-sum sequence T over G_0 with k(T) < 1.

Let $T_0 = \prod_{i=1}^{|G_0|} g_i^{l_i}$ be the minimal zero-sum sequence over G_0 such that $\mathsf{k}(T_0)$ is minimal. The minimality of G_0 implies that $l_i \ge 1$ for all $i \in [1, |G_0|]$. After renumbering if necessary, we let

$$\frac{\operatorname{ord}(g_1)}{l_1} = \min\{\frac{\operatorname{ord}(g_i)}{l_i} \mid i \in [1, |G_0|]\}.$$

By Lemma 2.2.3, $\left| \mathsf{L}\left(T_0^{\left\lceil \frac{\operatorname{ord}(g_1)}{l_1} \right\rceil}\right) \right| \ge 2$. If $T_0^{\left\lceil \frac{\operatorname{ord}(g_1)}{l_1} \right\rceil}$ divides $S^{\exp(G)}$, the assertion follows by Lemma 2.2.2. Suppose $T_0^{\left\lceil \frac{\operatorname{ord}(g_1)}{l_1} \right\rceil} \nmid S^{\exp(G)}$. Let

$$I = \{i \in [1, |G_0|] \mid \left\lceil \frac{\operatorname{ord}(g_1)}{l_1} \right\rceil l_i > \exp(G) \mathsf{v}_{g_i}(S)\}$$

Thus for each $i \in I$, we have

$$2\operatorname{ord}(g_i) > l_i \left\lceil \frac{\operatorname{ord}(g_i)}{l_i} \right\rceil \ge l_i \left\lceil \frac{\operatorname{ord}(g_1)}{l_1} \right\rceil > \exp(G)\mathsf{v}_{g_i}(S) \ge \exp(G),$$

which implies that $\operatorname{ord}(g_i) = \exp(G)$, $\mathsf{v}_{g_i}(S) = 1$, and $\left| \frac{\operatorname{ord}(g_1)}{l_1} \right| > \frac{\operatorname{ord}(g_i)}{l_i} = \frac{\exp(G)}{l_i}$.

Let $i_0 \in I$ such that $l_{i_0} = \max\{l_i \mid i \in I\}$. Therefore for every $j \in [1, |G_0|] \setminus I$, we have

$$l_j \le \frac{\exp(G)\mathsf{v}_{g_j}(S)}{\left\lceil \frac{\operatorname{ord}(g_1)}{l_1} \right\rceil} \le \frac{\exp(G)\mathsf{v}_{g_j}(S)}{\frac{\exp(G)}{l_{i_0}}} = l_{i_0}\mathsf{v}_{g_j}(S)$$

Note that for every $i \in I$, we have $l_i \leq l_{i_0} = l_{i_0} \mathsf{v}_{g_i}(S)$. It follows by $\mathsf{v}_{g_{i_0}}(T_0) = l_{i_0} = l_{i_0} \mathsf{v}_{g_{i_0}}(S) = \mathsf{v}_{g_{i_0}}(S^{l_{i_0}})$ that

 $S^{l_{i_0}} = T_0 \cdot W$, where W is a zero-sum sequence over $G_0 \setminus \{g_{i_0}\}$.

By the minimality of G_0 , we have $G_0 \setminus \{g_{i_0}\}$ is half-factorial which implies that $\mathsf{k}(W) \in \mathbb{N}$. Therefore $\mathsf{k}(T_0) = l_{i_0}\mathsf{k}(S) - \mathsf{k}(W)$ is an integer, a contradiction to $\mathsf{k}(T_0) < 1$.

4. Let $G_1 = \{g \in G_0 \mid \operatorname{ord}(g) = \exp(G) \vee_g(S)\}$. Suppose $G_0 \setminus G_1$ is not half-factorial. If $G_0 \setminus G_1$ is an LCN-set, then the assertion follows by Proposition 3.1.1 and Lemma 2.2.2. Otherwise there exits a minimal zero-sum sequence A over $G_0 \setminus G_1$ such that $\mathsf{k}(A) < 1$. We may assume that $\mathsf{k}(A)$ is minimal over all minimal zero-sum sequences over $G_0 \setminus G_1$ and that $\min\{\frac{\operatorname{ord}(g)}{\mathsf{v}_g(A)} \mid g \in \operatorname{supp}(A)\} = \frac{\operatorname{ord}(g_0)}{\mathsf{v}_{g_0}(A)}$ for some $g_0 \in \operatorname{supp}(A) \subset G_0 \setminus G_1$. Thus by Lemma 2.2.3, we have $|\mathsf{L}\left(A^{\left\lceil \frac{\operatorname{ord}(g_0)}{\mathsf{v}_{g_0}(A)} \right\rceil}\right)| \geq 2$. The definition of G_1 implies that $A^{\left\lceil \frac{\operatorname{ord}(g_0)}{\mathsf{v}_{g_0}(A)} \right\rceil}$ divides $S^{\exp(G)}$

and hence the assertion follows.

Suppose $G_0 \setminus G_1$ is half-factorial. Then G_1 is non-empty and hence $G_1 = \{g_0\}$ for some $g_0 \in G_0$. If G_0 is an LCN-set, then the assertion follows by Proposition 3.1.1 and Lemma 2.2.2. Otherwise there exits a minimal zero-sum sequence A over G_0 such that k(A) < 1. We may assume that k(A) is minimal over all minimal zero-sum sequences over G_0 and that $\min\{\frac{\operatorname{ord}(g)}{\mathsf{v}_g(A)} \mid g \in \operatorname{supp}(A)\} = \frac{\operatorname{ord}(g_1)}{\mathsf{v}_{g_1}(A)}$ for some $g_1 \in \operatorname{supp}(A) \subset G_0$. Thus by Lemma 2.2.3, we have $|\mathsf{L}\left(A^{\left\lceil \frac{\operatorname{ord}(g_1)}{\mathsf{v}_{g_1}(A)}\right\rceil}\right)| \ge 2$. For every $g \in G_0 \setminus G_1$, we obtain $\mathsf{v}_g(A) \left\lceil \frac{\operatorname{ord}(g_1)}{\mathsf{v}_{g_1}(A)} \right\rceil \le \mathsf{v}_g(A) \left\lceil \frac{\operatorname{ord}(g)}{\mathsf{v}_g(A)} \right\rceil < 2 \operatorname{ord}(g) \le \exp(G) \mathsf{v}_g(S)$. If $\mathsf{v}_{g_0}(A) \left\lceil \frac{\operatorname{ord}(g_1)}{\mathsf{v}_{g_1}(A)} \right\rceil \le \operatorname{ord}(g_0) = \exp(G)$, then

$$A^{\left|\frac{\operatorname{ord}(g_1)}{\operatorname{v}_{g_1}(A)}\right|}$$
 divides $S^{\exp(G)}$

and hence $|\mathsf{L}(S^{\exp(G)})| \geq 2$.

Otherwise for every $g \in G \setminus G_1$, we have

$$\frac{\exp(G)}{\mathsf{v}_{g_0}(A)} < \left\lceil \frac{\operatorname{ord}(g_1)}{\mathsf{v}_{g_1}(A)} \right\rceil \le \left\lceil \frac{\operatorname{ord}(g)}{\mathsf{v}_g(A)} \right\rceil \le \left\lceil \frac{\exp(G)\mathsf{v}_g(S)}{2\mathsf{v}_g(A)} \right\rceil \le \frac{\exp(G)\mathsf{v}_g(S)}{\mathsf{v}_g(A)}$$

Therefore $\mathsf{v}_g(A) < \mathsf{v}_{g_0}(A)\mathsf{v}_g(S)$ for all $g \in G_0 \setminus G_1$ which implies that A divides $S^{\mathsf{v}_{g_0}(A)}$. Thus there exits a zero-sum sequence W over $G_0 \setminus G_1$ such that $S^{\mathsf{v}_{g_0}(A)} = A \cdot W$. Since $G_0 \setminus G_1$ is half-factorial, we obtain $\mathsf{k}(A) = \mathsf{v}_{g_0}(A)\mathsf{k}(S) - \mathsf{k}(W)$ is an integer, a contradiction to $\mathsf{k}(A) < 1$.

Proof of Theorem 1.1. By the definition of transfer Krull monoid, it suffices to prove the assertions for $H = \mathcal{B}(G_0)$ and hence H is half-factorial if and only if G_0 is half-factorial. If G_0 is half-factorial, it is easy to see that $hf(G_0) = 1$ and $|\mathsf{L}(\prod_{g \in G_0} g^{2\operatorname{ord}(g)})| = 1$. Therefore we only need to show that (b) implies (c) and that (d) implies (c).

(b) \Rightarrow (c) Suppose $hf(G_0) = 1$ and assume to the contrary that G_0 is not half-factorial. Then there exists $A \in \mathcal{A}(G_0)$ such that $k(A) \neq 1$, whence supp(A) is not half-factorial. Therefore $hf(supp(A)) \geq 2$, a contradiction.

 $(\mathrm{d}) \Rightarrow (\mathrm{c}) \text{ Suppose } |\mathsf{L}(\prod_{g \in G_0} g^{2\operatorname{ord}(g)})| = 1 \text{ and assume to the contrary that } G_0 \text{ is not half-factorial. If } G_0 \text{ is an LCN set, then Proposition 3.1.1 implies that } |\mathsf{L}(\prod_{g \in G_0} g^{\operatorname{ord}(g)})| \ge 2, \text{ a contradiction. Thus there exists an atom } A \in \mathcal{A}(G_0) \text{ with } \mathsf{k}(A) < 1 \text{ and we may assume that } \mathsf{k}(A) \text{ is minimal over all atoms of } \mathcal{B}(G_0). \text{ Let } g_0 \in \operatorname{supp}(A). \text{ Then by Lemma 2.2.3, } we have } \left|\mathsf{L}\left(A^{\left\lceil \frac{\operatorname{ord}(g_0)}{v_{g_0}(A)}\right\rceil}\right)\right| \ge 2, \text{ a contradiction to } A^{\left\lceil \frac{\operatorname{ord}(g_0)}{v_{g_0}(A)}\right\rceil} | \prod_{g \in G_0} g^{2\operatorname{ord}(g)}.$

Proof of Theorem 1.2. By the definition of transfer Krull monoid, it sufficient to prove all assertions for $H = \mathcal{B}(G)$.

1. Suppose $\exp(G) < \infty$. If $|G| \ge 3$, then 2. implies that $hf(G) < \infty$. If $|G| \le 2$, then $\mathcal{B}(G)$ is half-factorial and hence hf(G) = 1.

Suppose $\exp(G) = \infty$. If there exists an element $g \in G$ with $\operatorname{ord}(g) = \infty$, then $A_n = ((n+1)g)(-ng)(-g)$ is an atom for every $n \in \mathbb{N}$. Since $\{(n+1)g, -ng, -g\}$ is not half-factorial and $|\mathsf{L}(A_n^n)| = 1$ for every $n \geq 2$, we obtain that $\mathsf{hf}(G) \geq n$ for every $n \geq 2$, that is, $\mathsf{hf}(G) = \infty$. Otherwise G is torsion. Then there exists a sequence $(g_i)_{i=1}^{\infty}$ with $g_i \in G$ and $\lim_{i\to\infty} \operatorname{ord}(g_i) = \infty$. It follows by 1. that $\mathsf{hf}(G) \geq \mathsf{hf}(\langle g_i \rangle) \geq \operatorname{ord}(g_i)$ for all $i \in \mathbb{N}$, that is, $\mathsf{hf}(G) = \infty$.

2. If G is an elementary 2-group and e_1, e_2 are two independent elements, then $\{e_1, e_2, e_1 + e_2\}$ is not a half-factorial set and $|\mathsf{L}(e_1e_2(e_1 + e_2))| = 1$ which implies that $\mathsf{hf}(G) \ge 2 = \exp(G)$. Otherwise there exists an element $g \in G$ with $\operatorname{ord}(g) = \exp(G) \ge 3$. Since $\{g, -g\}$ is not half-factorial and $|\mathsf{L}(g^{\operatorname{ord}(g)-1}(-g)^{\operatorname{ord}(g)-1})| = 1$, we obtain $\mathsf{hf}(G) \ge \operatorname{ord}(g) = \exp(G)$.

Let S be a zero-sum sequence over G such that $\operatorname{supp}(S)$ is not half-factorial. In order to prove $hf(G) \leq \left|\frac{3\exp(G)-3}{2}\right|$, we show that

$$\left|\mathsf{L}(S^{\left\lfloor\frac{3\exp(G)-3}{2}\right\rfloor})\right| \ge 2.$$

Set $G_0 = \operatorname{supp}(S)$. If G_0 is an LCN-set, the assertion follows by Proposition 3.1.1. Suppose there exists an atom $A \in \mathcal{A}(G_0)$ with $\mathsf{k}(A) < 1$. Let $A_0 \in \mathcal{A}(\operatorname{supp}(S))$ be such that $\mathsf{k}(A_0)$ is minimal over all minimal zero-sum sequences over G_0 and set $A_0 = g_1^{l_1} \cdot \ldots \cdot g_y^{l_y}$, where $y, l_1, \ldots l_y \in \mathbb{N}$ and $g_1, \ldots, g_y \in \operatorname{supp}(S)$ are pairwise distinct elements. If there exists $j \in [1, y]$ such that $2l_j \geq \operatorname{ord}(g_j)$, then $g_j^{\operatorname{ord}(g_i)}$ divides A_0^2 and hence $A_0^2 = g_j^{\operatorname{ord}(g_j)} \cdot W$ for some non-empty sequence $W \in \mathcal{B}(\operatorname{supp}(S))$. Thus $\mathsf{k}(W) = 2\mathsf{k}(A_0) - 1 < \mathsf{k}(A_0)$, a contradiction to the minimality of $\mathsf{k}(A_0)$. Therefore

$$2l_i \leq \operatorname{ord}(g_i) - 1$$
 for all $i \in [1, y]$.

After renumbering if necessary, we assume $\frac{\operatorname{ord}(g_1)}{l_1} = \min\{\frac{\operatorname{ord}(g_i)}{l_i} \mid i \in [1, y]\}$. Then

$$l_i \left\lceil \frac{\operatorname{ord}(g_1)}{l_1} \right\rceil \le l_i \left\lceil \frac{\operatorname{ord}(g_i)}{l_i} \right\rceil \le l_i \frac{\operatorname{ord}(g_i) + l_i - 1}{l_i} \le \operatorname{ord}(g_i) + \frac{\operatorname{ord}(g_i) - 1}{2} - 1 \le \frac{3 \exp(G) - 3}{2},$$

which implies $A_0^{\left\lfloor \frac{\operatorname{ord}(g_1)}{l_1} \right\rfloor}$ divides $S^{\left\lfloor \frac{3 \exp(G) - 3}{2} \right\rfloor}$. The assertion follows by Lemma 2.2.3.

3(a). Suppose that G is cyclic and that $g \in G$ with $\operatorname{ord}(g) = |G| \ge 3$. We will show that $hf(G) = \exp(G)$.

Let S be a zero-sum sequence over G such that $\operatorname{supp}(G)$ is not half-factorial. It suffices to show $|\mathsf{L}(S^{\exp(G)})| \geq 2$. If $|\{g \in \operatorname{supp}(S) \mid \operatorname{ord}(g) = |G|\mathsf{v}_g(S)\}| \leq 1$, then the assertion follows from Proposition 3.1.4. Suppose $|\{g \in \operatorname{supp}(S) \mid \operatorname{ord}(g) = |G|\mathsf{v}_g(S)\}| \geq 2$. Then there exist distinct $g_1, g_2 \in \operatorname{supp}(S)$ such that $\operatorname{ord}(g_1) = \operatorname{ord}(g_2) = |G|$. We may assume that $g_1 = kg_2$ for some $k \in \mathbb{N}_{\geq 2}$ with $\operatorname{gcd}(k, |G|) = 1$. It follows by $\mathsf{k}(g_1^{|G|-k} \cdot g_2) < 1$ that $G_0 = \{g_1, g_2\} = \{g_1, kg_1\}$ is not half-factorial. By Proposition 3.1.2, we obtain that $|\mathsf{L}(S^{\exp(G)})| \geq 2$.

3(b). Suppose G is a finite abelian group with $\exp(G) \leq 6$. We need to prove that $hf(G) = \exp(G)$. Let S be a zero-sum sequence over G such that $\sup(G)$ is not half-factorial. It suffices to show $|\mathsf{L}(S^{\exp(G)})| \geq 2$.

If supp(S) is an LCN-set, the assertion follows by Proposition 3.1.1. Thus there is a minimal zero-sum sequence W over supp(S) such that

k(W) < 1.

By Proposition 3.1.2 and Lemma 2.2.2, we have

$$|\operatorname{supp}(W)| \ge 3.$$

Suppose W | S. Since $\mathsf{k}(W) < 1$, it follows by Lemma 2.2 that $|\mathsf{L}(W^{\exp(G)})| \ge 2$ and hence $|\mathsf{L}(S^{\exp(G)})| \ge 2$. Therefore we may assume that $W \nmid S$, whence $|W| \ge |\operatorname{supp}(W)| + 1 \ge 4$. It follows that $6 \ge \exp(G) \ge \frac{|W|}{\mathsf{k}(W)} > |W| \ge 4$.

We distinguish two cases according to $\exp(G) \in \{5, 6\}$.

Case 1. $\exp(G) = 5$.

Then, $G \cong C_5^r$ and for all $W \in \mathcal{A}(\operatorname{supp}(S))$ with $\mathsf{k}(W) < 1$, we have that

 $W \nmid S$, $|\operatorname{supp}(W)| = 3$, and |W| = 4.

Let W_0 be an atom over supp(S) with $k(W_0) < 1$. Then W_0 and S must be of the forms

$$W_0 = g_1^2 g_2 g_3, \quad S = T g_1 g_2 g_3$$

where $g_1, g_2, g_3 \in \text{supp}(S)$ are pairwise distinct and $T \in \mathcal{F}(\text{supp}(S) \setminus \{g_1\})$ with $\sigma(T) = g_1$.

We may assume T is zero-sum free. Otherwise $T = T_0T'$ with T_0 is zero-sum and T' is zero-sum free and we can replace S by $T'g_1g_2g_3$, since $|\mathsf{L}((T'g_1g_2g_3)^5)| \ge 2$ implies that

 $|\mathsf{L}(S^5)| \ge 2$. Therefore S is a product of at most three atoms and every term of S has order 5.

Assume to the contrary that $|\mathsf{L}(S^5)| = 1$, i.e., $\mathsf{L}(S^5) = \{|T| + 3\}$. Since $g_1^5 g_2^5 g_3^5 = W_0^2(g_1g_2^3g_3^3)$ is a zero-sum subsequence of S^5 , we obtain that $g_1g_2^3g_3^3$ is an atom. Note that

$$(g_1^2 g_2 g_3)^2 S = (g_1^4 T) (g_1 g_2^3 g_3^3)$$
 is a zero-sum subsequence of S^5 .

Suppose S is an atom. Then $L(g_1^4T) = \{2\}$ and hence $|g_1^4T| \ge 2 \times 4 = 8$, i.e., $|T| \ge 4$. It follows by $\{5\} = L(S^5) = \{|T| + 3\}$ that |T| = 2, a contradiction.

Suppose S is a product of two atoms. Then $L(g_1^4T) = \{3\}$ and hence $|g_1^4T| \ge 3 \times 4 = 12$, i.e., $|T| \ge 8$. It follows by $\{10\} = L(S^5) = \{|T| + 3\}$ that |T| = 7, a contradiction.

Suppose S is a product of three atoms. Then $L(g_1^4T) = \{4\}$ which implies that $T = T_1T_2T_3T_4$ such that g_1T_i is zero-sum for all $i \in [1, 4]$. Since $g_1T_i | S$, we obtain $k(g_1T_i) \ge 1$ and hence $|g_1T_i| \ge 5$. Therefore $|g_1^4T| \ge 4 \times 5 = 20$, i.e., $|T| \ge 16$. It follows by $\{15\} = L(S^5) = \{|T| + 3\}$ that |T| = 12, a contradiction.

Case 2. $\exp(G) = 6$.

Let W be an atom over supp(S) with k(W) < 1. If |W| = 4, then |supp(W)| = 3 and hence W must be of the form

$$W = g_1^2 g_2 g_3$$

where $g_1, g_2, g_3 \in \text{supp}(S)$ are pairwise distinct. Since $(g_1^6)(g_2^6)(g_3^6) = W^3(g_2^3g_3^3)$, we obtain that $|\mathsf{L}(g_1^6g_2^6g_3^6)| \ge 2$. It follows by $g_1^6g_2^6g_3^6$ divides $S^{\exp(G)}$ that $|\mathsf{L}(S^{\exp(G)})| \ge 2$.

If |W| = 5, then |supp(W)| = 3 or 4 and hence W must be one of the following forms.

- i. $W = g_1^3 g_2 g_3$ with $g_1, g_2, g_3 \in \text{supp}(S)$ are pairwise distinct.
- ii. $W = g_1^2 g_2^2 g_3$ with $g_1, g_2, g_3 \in \text{supp}(S)$ are pairwise distinct.

iii. $W = g_1^2 g_2 g_3 g_4$ with $g_1, g_2, g_3, g_4 \in \text{supp}(S)$ are pairwise distinct.

Suppose (i) holds. Then $0 = 2\sigma(W) = 6g_1 + 2g_2 + 2g_3 = 2g_2 + 2g_3$. Since $(g_1^6)(g_2^6)(g_3^6) = W^2(g_2^2g_3^2)^2$, we obtain that $|\mathsf{L}(g_1^6g_2^6g_3^6)| \ge 2$. It follows by $g_1^6g_2^6g_3^6$ divides $S^{\exp(G)}$ that $|\mathsf{L}(S^{\exp(G)})| \ge 2$.

Suppose (ii) holds. Then $0 = 3\sigma(W) = 6g_1 + 6g_2 + 3g_3 = 3g_3$. Thus $\operatorname{ord}(g_3) = 2$ and hence $k(W) \ge 1/2 + 4/6 > 1$, a contradiction.

Suppose (iii) holds. Then $0 = 3\sigma(W) = 6g_1 + 3g_3 + 3g_3 + 3g_4 = 3g_2 + 3g_3 + 3g_4$. Therefore $W_0 = g_2^3 g_3^3 g_4^3$ is zero-sum. If $W_0 = g_1^6 g_2^6 g_3^6 g_4^6(W)^{-3}$ is not a minimal zero-sum sequence, then $|\mathsf{L}(g_1^6 g_2^6 g_3^6 g_4^6)| \ge 2$ and hence $|\mathsf{L}(S^{\exp(G)})| \ge 2$. If W_0 is minimal zero-sum, then $g_1^6 g_2^6 g_3^6 g_4^6 = (g_1^6) W_0^2$ implies that $|\mathsf{L}(g_1^6 g_2^6 g_3^6 g_4^6)| \ge 2$ and hence $|\mathsf{L}(S^{\exp(G)})| \ge 2$. \Box

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